Chapter 2: Counting Methods

Lesson 2.1: Counting Principles, page 73

1. a)

	khaki	black
red	red/khaki	red/black
blue	blue/khaki	blue/black
green	green/khaki	green/black

Each x represents a different combination. There are 6 x's; therefore, there are six different variations of the outfit to choose from.

b) The number of outfit variations, O, is related to the number of shirts and the number of shorts:

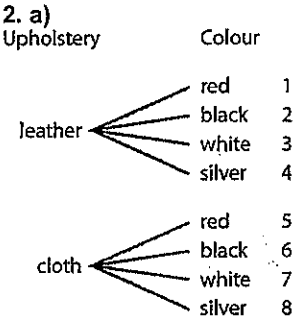
O = (number of shirts) · (number of shorts)

$O = 3 \cdot 2$

$O = 6$

There are six different variations of the outfit to choose from. This matches the part a) result.

2. a)



Therefore, there are 8 upholstery-colour choices that are available.

b) The number of upholstery-colour choices, U, is related to the number of colours and the number of kinds of upholstery:

U = (number of colours) · (number of upholstery)

$U = 4 \cdot 2$

$U = 8$

There are 8 upholstery-colour choices that are available. This matches the part a) result.

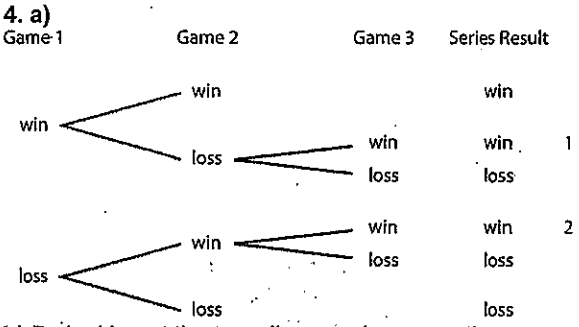
3. a) The Fundamental Counting Principle does not apply because tasks in this situation are related by the word OR.

b) The Fundamental Counting Principle does apply because tasks in this situation are related by the word AND.

c) The Fundamental Counting Principle does not apply because tasks in this situation are related by the word OR.

d) The Fundamental Counting Principle does apply because tasks in this situation are related by the word AND.

4. a)



b) By looking at the tree diagram, I can see there are 2 ways in which Kim's team can win the series despite losing one game.

5. The number of colour-size variations, C, is related to the number of colours and the number of sizes:

C = (number of colours) · (number of sizes)

$C = 5 \cdot 4$

$C = 20$

There are 20 colour-size variations that are available.

6. The number of computer systems, S, the employees can build for their customers is related to the number of desktop computers (dc), the number of monitors (m), the number of printers (p), and the number of software packages (sp):

S = (# of dc) · (# of m) · (# of p) · (# of sp)

$S = 5 \cdot 4 \cdot 6 \cdot 3$

$S = 360$

Therefore, the employees can build 360 different computer systems for their customers.

7. The number of possible meals, M, is related to the number of soups (s), the number of sandwiches (sw), the number of drinks (dr), and the number of desserts (d):

M = (# of s) · (# of sw) · (# of dr) · (# of d)

$M = 3 \cdot 5 \cdot 4 \cdot 2$

$M = 120$

Therefore, there are 120 different meal possibilities.

8. Event A: Selecting a rap CD

OR

Event B: Selecting a classic rock CD

$n(A \cup B) = n(A) + n(B)$

$n(A \cup B) = 8 + 10$

$n(A \cup B) = 18$

Therefore, Charlene can select from 18 CDs to play in her car stereo that will match Tom's musical tastes.

9. a) The number of different PIN combinations, C, is related to the number of digits from which to select for each digit of the PIN, P:

$C = P_1 \cdot P_2 \cdot P_3 \cdot P_4 \cdot P_5$

$C = 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$

$C = 59\,049$

There are 59 049 different five-digit PIN combinations.

b) The number of different PIN combinations, N, is related to the number of digits from which to select for each digit of the PIN, D:
$N = D_1 \cdot D_2 \cdot D_3 \cdot D_4 \cdot D_5$
$N = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$
$N = 15\,120$
There are only 15 120 different five-digit PIN combinations in which the digits cannot repeat.

10. The number of different bytes that can be created, N, is related to the number of digits from which to choose for each digit of the byte, B:
$N = B_1 \cdot B_2 \cdot B_3 \cdot B_4 \cdot B_5 \cdot B_6 \cdot B_7 \cdot B_8$
$N = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
$N = 256$
Therefore, 256 different bytes can be created.

11. a) The number of different upper-case letter possibilities, N, is related to the number of upper-case letters from which to choose for each odd position of the country's postal code, P:
$N = P_1 \cdot P_3 \cdot P_5$
$N = 26 \cdot 26 \cdot 26$
$N = 17\,576$
The number of different digit possibilities, D, is related to the number of digits from which to choose for each even position of the country's postal code, P:
$D = P_2 \cdot P_4 \cdot P_6$
$D = 10 \cdot 10 \cdot 10$
$D = 1000$
The number of different postal codes that are possible in this country, C, is related to the number of upper-case letter possibilities, N, and the number of digit possibilities, D:
$C = N \cdot D$
$C = 17\,576 \cdot 1000$
$C = 17\,576\,000$
Therefore, 17 576 000 postal codes are possible.
b) The number of different upper-case letter possibilities, N, is related to the number of upper-case letters from which to choose for each odd position of the country's postal code, P:
$N = P_1 \cdot P_3 \cdot P_5$
$N = 21 \cdot 21 \cdot 21$
$N = 9261$
The number of different digit possibilities, D, remains the same since all digits can be used. The number of different postal codes that are possible in Canada, C, is related to the number of uppercase letter possibilities, N, and the number of digit possibilities, D:
$C = N \cdot D$
$C = 9261 \cdot 1000$
$C = 9\,261\,000$
Therefore, 9 261 000 postal codes are possible in Canada.

12. To answer this question, I need to determine how many digit combinations there are for the last four digits of one of these two phone numbers, and then multiply it by 2.
The number of digit combinations, C, is related to the number of possible digits for each of the last four digits of one of the phone numbers, P:
$C = P_1 \cdot P_2 \cdot P_3 \cdot P_4$
$C = 10 \cdot 10 \cdot 10 \cdot 10$
$C = 10\,000$
The number of phone numbers is $2C$ since there are two given templates for the phone numbers in the question.
$2C = 2(10\,000)$
$2C = 20\,000$
Therefore, 20 000 different phone numbers are possible for this town.

13. The number of different codes, C, is related to number of positions from which to select for each switch of the garage door opener, G:
$C = G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_5 \cdot G_6 \cdot G_7 \cdot G_8 \cdot G_9$
$C = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
$C = 19\,683$
Therefore, 19 683 different codes are possible.

14. Event A: Selecting a pickup truck OR
Event B: Selecting a passenger van OR
Event C: Selecting a car OR
Event D: Selecting a sports utility vehicle
$n(A \cup B \cup C \cup D) = n(A) + n(B) + n(C) + n(D)$
$n(A \cup B \cup C \cup D) = 8 + 10 + 35 + 12$
$n(A \cup B \cup C \cup D) = 65$
Therefore, a customer has 65 choices when renting just one vehicle.

15. a) Multiply the number of sizes of the crust, by the number of types of the crust, by the number of types of cheese, by the number of types of tomato sauce.
$2 \cdot 2 \cdot 2 \cdot 2 = 16$
Multiply this number by the number of different toppings.
$16 \cdot 20 = 320$
Therefore, there are 320 different pizzas that can be made with any crust, cheese, tomato sauce, and 1 topping.
b) Multiply the number of types of cheese by the number of types of tomato sauce.
$2 \cdot 2 = 4$
Therefore, there are 4 different pizzas that can be made with a thin whole-wheat crust, tomato sauce, cheese, and no toppings.

16. a) The number of different upper-case letter possibilities, N, is related to the number of upper-case letters from which to choose for each of the first three positions of the Alberta licence plate, P:

$N = P_1 \cdot P_2 \cdot P_3$

$N = 24 \cdot 24 \cdot 24$

$N = 13\,824$

The number of different digit possibilities, D, is related to the number of digits from which to choose for each of the last three positions of the Alberta licence plate, P:

$D = P_4 \cdot P_5 \cdot P_6$

$D = 10 \cdot 10 \cdot 10$

$D = 1000$

The number of different possible Alberta licence plates, C, is related to the number of upper-case letter possibilities, N, and the number of digit possibilities, D:

$C = N \cdot D$

$C = 13\,824 \cdot 1000$

$C = 13\,824\,000$

So, 13 824 000 Alberta licence plates are possible.

b) The number of different upper-case letter possibilities, N, remains the same since the number of letters in the plates and the number of letters that can be used is the same as in a).

The number of different digit possibilities, D, is related to the number of digits from which to choose for each of the last four positions of the Alberta licence plate, P:

$D = P_4 \cdot P_5 \cdot P_6 \cdot P_7$

$D = 10 \cdot 10 \cdot 10 \cdot 10$

$D = 10\,000$

The number of different possible Alberta licence plates, C, is related to the number of upper-case letter possibilities, N, and the number of digit possibilities, D:

$C = N \cdot D$

$C = 13\,824 \cdot 10\,000$

$C = 138\,240\,000$

$138\,240\,000 - 13\,824\,000 = 124\,416\,000$

So, 124 416 000 more licence plates are possible.

17. e.g., If multiple tasks are related by AND, it means the Fundamental Counting Principle can be used and the total number of solutions is the product of the solutions to each task. For example: A 4-digit PIN involves choosing the 1st digit AND the 2nd digit AND the 3rd digit AND the 4th digit. So the number of solutions is $10 \cdot 10 \cdot 10 \cdot 10 = 10\,000$. OR means the solution must meet at least one condition so you must add the number of solutions to each condition, and then subtract the number of solutions that meet all conditions. For example: Calculating the number of 4-digit PINs that start with 3 OR end with 3. The solution is the number of PINs that start with 3, plus the number of PINs that end with 3, minus the number of PINs that both start and end with 3:

$1000 + 1000 - 100 = 1900$.

18. a) i) Event A: Drawing a king OR
Event B: Drawing a queen

$n(A \cup B) = n(A) + n(B)$

$n(A \cup B) = 4 + 4$

$n(A \cup B) = 8$

$\text{Likelihood} = \frac{8}{52}$

$\text{Likelihood} = \frac{2}{13}$

Therefore, there is a 2 in 13 chance that a king or a queen will be drawn.

ii) Event A: Drawing a diamond
OR
Event B: Drawing a club

$n(A \cup B) = n(A) + n(B)$

$n(A \cup B) = 13 + 13$

$n(A \cup B) = 26$

$\text{Likelihood} = \frac{26}{52}$

$\text{Likelihood} = \frac{1}{2}$

Therefore, there is a 1 in 2 chance that a diamond or a club will be drawn.

iii) Event A: Drawing an Ace OR
Event B: Drawing a spade

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cup B) = 4 + 13 - 1$

$n(A \cup B) = 16$

$\text{Likelihood} = \frac{16}{52}$

$\text{Likelihood} = \frac{4}{13}$

Therefore, there is a 4 in 13 chance that an ace or a spade will be drawn.

b) No, e.g., because the Fundamental Counting Principle only applies when tasks are related by the word AND.

19. e.g., To begin, there are 90 two-digit numbers. There are 10 with a 1 in the tens column, 10 with a 2 in the tens column, and this pattern continues until I reach the 10 with a 9 in the tens column. Next, I must determine the numbers that are divisible by either 2 or 5. I know that every other number is even and thus divisible by 2. This means that $\frac{90}{2}$ or 45 of the two-digit numbers are divisible by 2. The two-digit numbers that are divisible by 5 can be found by starting at the first two-digit number, 10, and then counting by 5 until I get to a three-digit number. By doing this, I can determine that the two-digit numbers that are divisible by 5 are: 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95. There are 18 of them. I see that half of them or 9 are even and thus divisible by 2.

Therefore, there are 9 numbers that are divisible by 5 and not by 2. If I add this together with the number of two-digit numbers that are divisible by two (45), I see that there are 54 two-digit numbers divisible by 2 or 5. Whatever is leftover from the two digit numbers are the ones that are not divisible by either 2 or 5. This amount is:
$90 - 54 = 36$. Thus, there are 36 two-digit numbers that are not divisible by either 2 or 5.

20. The number of different outcomes for a student's test, N, is related to the number of possible answers for each question on the test, T:
$N = T_1 \cdot T_2 \cdot T_3 \cdot T_4 \cdot T_5 \cdot T_6 \cdot T_7 \cdot T_8 \cdot T_9 \cdot T_{10}$
$N = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
$N = 1024$
A perfect score is only 1 out of these 1024 outcomes; therefore, there is a 1 in 1024 chance that the student will get a perfect score.

21. This question is solved by constant application of the Fundamental Counting Principle.
If an item from each category is selected:
$O = 3 \cdot 5 \cdot 4 \cdot 2$
$O = 120$
If no soup is selected:
$O = 5 \cdot 4 \cdot 2$
$O = 40$
If no sandwich is selected:
$O = 3 \cdot 4 \cdot 2$
$O = 24$
If no drink is selected:
$O = 3 \cdot 5 \cdot 2$
$O = 30$
If no dessert is selected:
$O = 3 \cdot 5 \cdot 4$
$O = 60$
If no soup or sandwich is selected:
$O = 4 \cdot 2$
$O = 8$
If no soup or drink selected:
$O = 5 \cdot 2$
$O = 10$
If no soup or dessert is selected:
$O = 5 \cdot 4$
$O = 20$
If no sandwich or drink is selected:
$O = 3 \cdot 2$
$O = 6$
If no sandwich or dessert is selected:
$O = 3 \cdot 4$
$O = 12$
If no drink or dessert is selected:
$O = 3 \cdot 5$
$O = 15$

If only a soup, sandwich, drink or dessert is selected:
$O = 3, 5, 4, 2$
$T_{otal} = 120 + 40 + 24 + 30 + 60 + 8 + 10 + 20 + 6 + 12 + 15 + 3 + 5 + 4 + 2$
$T_{otal} = 359$
Therefore, 359 meals are possible if you do not have to choose an item from a category.

Lesson 2.2: Introducing Permutations and Factorial Notation, page 81

1. a) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$6! = 720$

b) $9 \cdot 8! = 9 \cdot (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
$9 \cdot 8! = 9 \cdot 40320$
$9 \cdot 8! = 362880$

c) $\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$
$\frac{5!}{3!} = 5 \cdot 4 \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$
$\frac{5!}{3!} = 5 \cdot 4 \cdot \frac{3!}{3!}$
$\frac{5!}{3!} = 5 \cdot 4 \cdot 1$
$\frac{5!}{3!} = 20$

d) $\frac{8!}{7!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$
$\frac{8!}{7!} = 8 \cdot \frac{7}{7} \cdot \frac{6}{6} \cdot \frac{5}{5} \cdot \frac{4}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$
$\frac{8!}{7!} = 8 \cdot \frac{7!}{7!}$
$\frac{8!}{7!} = 8 \cdot 1$
$\frac{8!}{7!} = 8$

e) $3! \cdot 2! = (3 \cdot 2 \cdot 1) \cdot (2 \cdot 1)$
$3! \cdot 2! = 6 \cdot 2$
$3! \cdot 2! = 12$

f) $\frac{9!}{4!3!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1) \cdot (3 \cdot 2 \cdot 1)}$
$\frac{9!}{4!3!} = \frac{9}{3} \cdot \frac{8}{2} \cdot \frac{7}{1} \cdot 6 \cdot 5 \cdot \frac{4}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$
$\frac{9!}{4!3!} = \frac{9}{3} \cdot \frac{8}{2} \cdot 7 \cdot 6 \cdot 5 \cdot \frac{4!}{4!}$
$\frac{9!}{4!3!} = \frac{9}{3} \cdot \frac{8}{2} \cdot 7 \cdot 6 \cdot 5 \cdot 1$
$\frac{9!}{4!3!} = 3 \cdot 4 \cdot 7 \cdot 6 \cdot 5$
$\frac{9!}{4!3!} = 2520$

2. a) e.g., There are six permutations of Ken, Sarah, and Raj. I figured this out by making a table showing each permutation and the three possible positions. Also, $3 \cdot 2 \cdot 1 = 6$

	Position 1	Position 2	Position 3
Permutation 1	Ken	Sarah	Raj
Permutation 2	Ken	Raj	Sarah
Permutation 3	Sarah	Ken	Raj
Permutation 4	Sarah	Raj	Ken
Permutation 5	Raj	Ken	Sarah
Permutation 6	Raj	Sarah	Ken

b) Let L represent the total number of permutations:

$L = 3 \cdot 2 \cdot 1$

$L = 3!$

3. a) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

b) $9 \cdot 8 \cdot 7 = \dfrac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$9 \cdot 8 \cdot 7 = \dfrac{9!}{6!}$

c) $\dfrac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} = \dfrac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \dfrac{12!}{12!}$

$\dfrac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} = \dfrac{15!}{12!4!}$

d) $100 \cdot 99 = 100 \cdot 99 \cdot \dfrac{98!}{98!}$

$100 \cdot 99 = \dfrac{100!}{98!}$

4. Expressions a), c), and d) are undefined because factorial notation is only defined for natural numbers.

5. a) $8 \cdot 7 \cdot 6! = 8 \cdot 7 \cdot (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

$8 \cdot 7 \cdot 6! = 8 \cdot 7 \cdot 720$

$8 \cdot 7 \cdot 6! = 56 \cdot 720$

$8 \cdot 7 \cdot 6! = 40320$

b) $\dfrac{12!}{10!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$\dfrac{12!}{10!} = 12 \cdot 11 \cdot \dfrac{10}{10} \cdot \dfrac{9}{9} \cdot \dfrac{8}{8} \cdot \dfrac{7}{7} \cdot \dfrac{6}{6} \cdot \dfrac{5}{5} \cdot \dfrac{4}{4} \cdot \dfrac{3}{3} \cdot \dfrac{2}{2} \cdot \dfrac{1}{1}$

$\dfrac{12!}{10!} = 12 \cdot 11 \cdot \dfrac{10!}{10!}$

$\dfrac{12!}{10!} = 12 \cdot 11 \cdot 1$

$\dfrac{12!}{10!} = 132$

c) $\dfrac{8!}{2! \cdot 6!} = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}$

$\dfrac{8!}{2! \cdot 6!} = \dfrac{8}{2} \cdot \dfrac{7}{1} \cdot \dfrac{6}{6} \cdot \dfrac{5}{5} \cdot \dfrac{4}{4} \cdot \dfrac{3}{3} \cdot \dfrac{2}{2} \cdot \dfrac{1}{1}$

$\dfrac{8!}{2! \cdot 6!} = \dfrac{8}{2} \cdot 7 \cdot \dfrac{6!}{6!}$

$\dfrac{8!}{2! \cdot 6!} = \dfrac{8}{2} \cdot 7 \cdot 1$

$\dfrac{8!}{2! \cdot 6!} = 4 \cdot 7$

$\dfrac{8!}{2! \cdot 6!} = 28$

d) $\dfrac{7 \cdot 6!}{5!} = \dfrac{7 \cdot (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$\dfrac{7 \cdot 6!}{5!} = 7 \cdot 6 \cdot \dfrac{5}{5} \cdot \dfrac{4}{4} \cdot \dfrac{3}{3} \cdot \dfrac{2}{2} \cdot \dfrac{1}{1}$

$\dfrac{7 \cdot 6!}{5!} = 7 \cdot 6 \cdot \dfrac{5!}{5!}$

$\dfrac{7 \cdot 6!}{5!} = 7 \cdot 6 \cdot 1$

$\dfrac{7 \cdot 6!}{5!} = 42$

e) $4\left(\dfrac{6!}{2! \cdot 2!}\right) = 4\left[\dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot (2 \cdot 1)}\right]$

$4\left(\dfrac{6!}{2! \cdot 2!}\right) = 4\left(\dfrac{6}{2} \cdot \dfrac{5}{1} \cdot 4 \cdot 3 \cdot \dfrac{2}{2} \cdot \dfrac{1}{1}\right)$

$4\left(\dfrac{6!}{2! \cdot 2!}\right) = 4\left(\dfrac{6}{2} \cdot 5 \cdot 4 \cdot 3 \cdot \dfrac{2!}{2!}\right)$

$4\left(\dfrac{6!}{2! \cdot 2!}\right) = 4\left(\dfrac{6}{2} \cdot 5 \cdot 4 \cdot 3 \cdot 1\right)$

$4\left(\dfrac{6!}{2! \cdot 2!}\right) = 4(3 \cdot 5 \cdot 4 \cdot 3)$

$4\left(\dfrac{6!}{2! \cdot 2!}\right) = 4(180)$

$4\left(\dfrac{6!}{2! \cdot 2!}\right) = 720$

f) $4! + 3! + 2! + 1! = (4 \cdot 3 \cdot 2 \cdot 1) + (3 \cdot 2 \cdot 1) + (2 \cdot 1) + 1$

$4! + 3! + 2! + 1! = 24 + 6 + 2 + 1$

$4! + 3! + 2! + 1! = 33$

6. a) $\dfrac{n!}{(n-1)!} = \dfrac{(n)(n-1)(n-2)(n-3)\ldots(3)(2)(1)}{(n-1)(n-2)(n-3)\ldots(3)(2)(1)}$

$\dfrac{n!}{(n-1)!} = \dfrac{(n)(n-1)!}{(n-1)!}$

$\dfrac{n!}{(n-1)!} = n$

b) $(n+4)(n+3)(n+2)!$

$=(n+4)(n+3)[(n+2)(n+1)(n)(n-1)...(3)(2)(1)]$

$=(n+4)(n+3)(n+2)(n+1)(n)(n-1)...(3)(2)(1)$

$=(n+4)!$

c) $\frac{(n+1)!}{n!}=\frac{(n+1)(n)(n-1)(n-2)...(3)(2)(1)}{(n)(n-1)(n-2)...(3)(2)(1)}$

$\frac{(n+1)!}{n!}=\frac{(n+1)(n!)}{n!}$

$\frac{(n+1)!}{n!}=n+1$

d)

$\frac{n!}{(n-3)!}=\frac{(n)(n-1)(n-2)(n-3)(n-4)(n-5)...(3)(2)(1)}{(n-3)(n-4)(n-5)...(3)(2)(1)}$

$\frac{n!}{(n-3)!}=\frac{(n)(n-1)(n-2)(n-3)!}{(n-3)!}$

$\frac{n!}{(n-3)!}=(n)(n-1)(n-2)$

$\frac{n!}{(n-3)!}=(n^2-n)(n-2)$

$\frac{n!}{(n-3)!}=n^3-3n^2+2n$

e) $\frac{(n+5)!}{(n+3)!}=\frac{(n+5)(n+4)(n+3)(n+2)(n+1)...(3)(2)(1)}{(n+3)(n+2)(n+1)...(3)(2)(1)}$

$\frac{(n+5)!}{(n+3)!}=\frac{(n+5)(n+4)(n+3)!}{(n+3)!}$

$\frac{(n+5)!}{(n+3)!}=(n+5)(n+4)$

$\frac{(n+5)!}{(n+3)!}=n^2+9n+20$

f) $\frac{(n-2)!}{(n-1)!}=\frac{(n-2)(n-3)(n-4)...(3)(2)(1)}{(n-1)(n-2)(n-3)(n-4)...(3)(2)(1)}$

$\frac{(n-2)!}{(n-1)!}=\frac{(n-2)!}{(n-1)(n-2)!}$

$\frac{(n-2)!}{(n-1)!}=\frac{1}{n-1}$

7. There are nine students in the lineup, so there are nine possible positions. Let L represent the total number of permutations:

$L = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$L = 9!$

$L = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$L = 72 \cdot 7 \cdot 30 \cdot 4 \cdot 6$

$L = 72 \cdot 210 \cdot 24$

$L = 362\,880$

There are 362 880 permutations for the nine students at the Calgary Stampede.

8. There are five students in the club and there are five possible positions. Let L represent the total number of permutations:

$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$L = 5!$

$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$L = 20 \cdot 6$

$L = 120$

There are 120 different ways to select members for the five positions.

9. There are six activities to do and there are six days. Let L represent the total number of permutations:

$L = 6!$

$L = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$L = 30 \cdot 4 \cdot 6$

$L = 120 \cdot 6$

$L = 720$

There are 720 different ways they can sequence these activities over the six days.

10. There are 28 movies, so there are 28 possible spots for the movies to go. Let L represent the total number of permutations:

$L = 28!$

$L = 3.048... \times 10^{29}$

There are about 3.05×10^{29} possible permutations of the movie list.

11. a)

$\frac{(n+1)!}{n!}=10$

$\frac{(n+1)(n)(n-1)...(3)(2)(1)}{(n)(n-1)...(3)(2)(1)}=10$

$\frac{(n+1)(n!)}{n!}=10$

$n+1=10$

$n=9$

Check $n = 9$

LS	RS
$\frac{(9+1)!}{9!}$	10
$\frac{10!}{9!}$	
$\frac{10 \cdot 9!}{9!}$	
10	

There is one solution, $n = 9$.

b) $\dfrac{(n+2)!}{n!}=6$

$$\frac{(n+2)(n+1)(n)(n-1)\ldots(3)(2)(1)}{(n)(n-1)\ldots(3)(2)(1)}=6$$

$$\frac{(n+2)(n+1)(n!)}{n!}=6$$

$$(n+2)(n+1)=6$$

$$n^2+n+2n+2=6$$

$$n^2+3n+2=6$$

$$n^2+3n-4=0$$

$$(n+4)(n-1)=0$$

$n+4=0$ or $n-1=0$

$n=-4$ $\quad$ $n=1$

Check $n=-4$

LS	RS
$\dfrac{(-4+2)!}{(-4)!}$	6
$\dfrac{(-2)!}{(-4)!}$ is undefined	

Check $n=1$

LS	RS
$\dfrac{(1+2)!}{1!}$	6
$\dfrac{3!}{1!}$	
$\dfrac{3\cdot2\cdot1!}{1!}$	
$3\cdot2$	
6	

There is one solution, $n=1$.

c) $\dfrac{(n-1)!}{(n-2)!}=8$

$$\frac{(n-1)(n-2)(n-3)\ldots(3)(2)(1)}{(n-2)(n-3)\ldots(3)(2)(1)}=8$$

$$\frac{(n-1)(n-2)!}{(n-2)!}=8$$

$$n-1=8$$

$$n=9$$

Check $n=9$

LS	RS
$\dfrac{(9-1)!}{(9-2)!}$	8
$\dfrac{8!}{7!}$	
$\dfrac{8\cdot7!}{7!}$	
8	

There is one solution, $n=9$.

d) $\dfrac{3(n+1)!}{(n-1)!}=126$

$$\frac{3(n+1)(n)(n-1)(n-2)\ldots(3)(2)(1)}{(n-1)(n-2)\ldots(3)(2)(1)}=126$$

$$\frac{3(n+1)(n)(n-1)!}{(n-1)!}=126$$

$$3(n+1)(n)=126$$

$$3(n^2+n)=126$$

$$3(n^2+n)-126=0$$

$$3\left[(n^2+n)-42\right]=0$$

$$3(n^2+n-42)=0$$

$$3(n+7)(n-6)=0$$

$n+7=0$ or $n-6=0$

$n=-7$ $\quad$ $n=6$

Check $n=-7$

LS	RS
$\dfrac{3(-7+1)!}{(-7-1)!}$	126
$\dfrac{3(-6)!}{(-8)!}$ is undefined	

Check $n=6$

LS	RS
$\dfrac{3(6+1)!}{(6-1)!}$	126
$\dfrac{3(7!)}{5!}$	
$\dfrac{3\cdot(7\cdot6\cdot5!)}{5!}$	
$3\cdot7\cdot6$	
126	

There is one solution, $n=6$.

12. There are eight more players left to organize so there are eight more spots left in the batting order. Let L represent the number of permutations:
$L = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 8!$
$L = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 8 \cdot 42 \cdot 20 \cdot 6$
$L = 336 \cdot 120$
$L = 40\ 320$
There are 40 320 possible batting orders.

13. There are 7 possible digits to use and there are 7 digits in each serial number. Let L represent the number of permutations:
$L = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 7!$
There are 7! possible serial numbers. This makes sense because, e.g., the integer in the factorial (7 in this case) for the number of permutations is normally equal to the number of spots in which there are things to place. There are seven spots in the serial number so this means that the number of permutations should be 7! which matches the answer that was found.

14. There are 5 cars to be arranged between the engine and the caboose so there are 5 spots in which the cars can be lined up. Let L represent the number of permutations:
$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 5!$
$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 20 \cdot 6$
$L = 120$
There are 120 ways for the cars to be arranged between the engine and the caboose.

15. There would be 7 chuckwagons behind Brant's so there are 7 spots where the other drivers could finish. Let L represent the number of permutations:
$L = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 7!$
$L = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 42 \cdot 20 \cdot 6$
$L = 42 \cdot 120$
$L = 5040$
If Brant's wagon wins, there are 5040 different orders in which the eight chuckwagons can finish.

16. a) e.g., YKONU, YUKNO, YKNOU
b) There are five letters so there are five spots to put the letters. Let L represent the number of permutations:
$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 5!$
There are 5! possible permutations. This makes sense because e.g., the integer in the factorial (5 in this case) for the number of permutations is normally equal to the number of spots in which there are things to place. There are five spots to place the letters so this means that the number of permutations should be 5! which matches the answer that was found.

17. a) e.g., Using trial and error, I have the following calculations:
$1! = 1$, $2^1 = 2$; $2! = 2$, $2^2 = 4$;
$3! = 6$, $2^3 = 8$; $4! = 24$, $2^4 = 16$
I notice that for $n = 4$, $n!$ is greater than 2^n. This continues for $n \geq 4$ because 2^4 will keep getting multiplied by 2, while 4! will keep getting multiplied by numbers greater than 2 to obtain the higher factorials.
b) e.g., Using what I have in a), I know that for $n < 4$, $n!$ is not greater than 2^n. The calculations for these values of n are shown in a). Thus for $n = \{1, 2, 3\}$, $n!$ is less than 2^n.

18. e.g., First, figure out how many ways Darlene and Arnold can be placed next to each other in the line. This can be found using a table.

Darlene	Arnold
1	2
2	3
3	4
4	5
5	6
6	7
7	8
8	9
9	10
2	1
3	2
4	3
5	4
6	5
7	6
8	7
9	8
10	9

From the table I can see that there are 18 different ways for Darlene and Arnold to be placed next to each other in the line. For every one of those 18 ways, there are 8 other dancers to be placed in 8 different spots in the line. Let L represent the number of permutations:
$L = 18(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
$L = 18(8!)$
$L = 18(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
$L = 18(8 \cdot 42 \cdot 20 \cdot 6)$
$L = 18(336 \cdot 120)$
$L = 18(40\ 320)$
$L = 725\ 760$
There are 725 760 possible arrangements of the dancers for the Red River Jig.

Lesson 2.3: Permutations When All Objects Are Distinguishable, page 93

1. a) $_5P_2 = \frac{5!}{(5-2)!}$

$_5P_2 = \frac{5!}{3!}$

$_5P_2 = \frac{5 \cdot 4 \cdot 3!}{3!}$

$_5P_2 = 5 \cdot 4$

$_5P_2 = 20$

b) $_8P_6 = \frac{8!}{(8-6)!}$

$_8P_6 = \frac{8!}{2!}$

$_8P_6 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!}$

$_8P_6 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$

$_8P_6 = 20160$

c) $_{10}P_5 = \frac{10!}{(10-5)!}$

$_{10}P_5 = \frac{10!}{5!}$

$_{10}P_5 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5!}$

$_{10}P_5 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$

$_{10}P_5 = 30240$

d) $_9P_0 = \frac{9!}{(9-0)!}$

$_9P_0 = \frac{9!}{9!}$

$_9P_0 = 1$

e) $_7P_7 = \frac{7!}{(7-7)!}$

$_7P_7 = \frac{7!}{0!}$

$_7P_7 = \frac{7!}{1}$

$_7P_7 = 7!$

$_7P_7 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$_7P_7 = 5040$

f) $_{15}P_5 = \frac{15!}{(15-5)!}$

$_{15}P_5 = \frac{15!}{10!}$

$_{15}P_5 = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10!}{10!}$

$_{15}P_5 = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11$

$_{15}P_5 = 360360$

2. a) e.g.,

Permutation	President	Vice-President
1	Katrina	Jess
2	Katrina	Nazir
3	Katrina	Mohamad
4	Jess	Katrina
5	Jess	Nazir
6	Jess	Mohamad
7	Nazir	Jess
8	Nazir	Katrina
9	Nazir	Mohamad
10	Mohamad	Nazir
11	Mohamad	Jess
12	Mohamad	Katrina

There are 12 different ways that a president and vice-president can be elected.

b) $_nP_r = \frac{n!}{(n-r)!}$

$_4P_2 = \frac{4!}{(4-2)!}$

$_4P_2 = \frac{4!}{2!}$

$_4P_2 = \frac{4 \cdot 3 \cdot 2!}{2!}$

$_4P_2 = 4 \cdot 3$

$_4P_2 = 12$

The formula for $_nP_r$ gives an answer of 12. This matches my results from part a).

3. a) $_nP_r = \frac{n!}{(n-r)!}$

$_6P_4 = \frac{6!}{(6-4)!}$

$_6P_4 = \frac{6!}{2!}$

$_6P_4 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!}$

$_6P_4 = 6 \cdot 5 \cdot 4 \cdot 3$

$_6P_4 = 360$

There are 360 different ways the chocolate bars can be distributed.

b) $_nP_r = \frac{n!}{(n-r)!}$

$_6P_1 = \frac{6!}{(6-1)!}$

$_6P_1 = \frac{6!}{5!}$

$_6P_1 = \frac{6 \cdot 5!}{5!}$

$_6P_1 = 6$

The chocolate bars can be distributed in 6 different ways.

4. ${}_{10}P_8$ is larger. e.g., I know this by looking at the formula for ${}_nP_r$. The numerator is the same for both values since n is the same. The denominator will be smaller for the first value since it has a greater r. When you divide a numerator by two different denominators, the final value is greater for the one with the smaller denominator. Based on this, I know that ${}_{10}P_8$ has the larger value since its expansion has the smaller denominator.

5. ${}_9P_3 = \frac{9!}{(9-3)!}$

${}_9P_3 = \frac{9!}{6!}$

${}_9P_3 = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6!}$

${}_9P_3 = 9 \cdot 8 \cdot 7$

${}_9P_3 = 504$

There are 504 different ways the positions can be filled.

6. ${}_{15}P_4 = \frac{15!}{(15-4)!}$

${}_{15}P_4 = \frac{15!}{11!}$

${}_{15}P_4 = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11!}{11!}$

${}_{15}P_4 = 15 \cdot 14 \cdot 13 \cdot 12$

${}_{15}P_4 = 32760$

There are 32 760 possible executive committees.

7. ${}_8P_8 = \frac{8!}{(8-8)!}$

${}_8P_8 = \frac{8!}{0!}$

${}_8P_8 = \frac{8!}{1}$

${}_8P_8 = 8!$

${}_8P_8 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

${}_8P_8 = 40320$

Therefore, 40 320 different signals could be created.

8. ${}_{5000}P_3 = \frac{5000!}{(5000-3)!}$

${}_{5000}P_3 = \frac{5000!}{4997!}$

${}_{5000}P_3 = \frac{5000 \cdot 4999 \cdot 4998 \cdot 4997!}{4997!}$

${}_{5000}P_3 = 5000 \cdot 4999 \cdot 4998$

${}_{5000}P_3 = 124\ 925\ 010\ 000$

There are about 124 925 010 000 different ways the tickets can be drawn.

9. e.g., Assuming that any of the 10 digits can be put in any of the 8 remaining spots for the SINs, let S represent the number of social insurance numbers:

$S = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

$S = 10^8$

$S = 100\ 000\ 000$

There are 100 000 000 different SINs that can be registered in each of these groups of provinces and territories.

10. a) ${}_{12}P_5 = \frac{12!}{(12-5)!}$

${}_{12}P_5 = \frac{12!}{7!}$

${}_{12}P_5 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{7!}$

${}_{12}P_5 = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$

${}_{12}P_5 = 95040$

There are 95 040 ways the coach can select the starting five players.

b) ${}_{11}P_4 = \frac{11!}{(11-4)!}$

${}_{11}P_4 = \frac{11!}{7!}$

${}_{11}P_4 = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{7!}$

${}_{11}P_4 = 11 \cdot 10 \cdot 9 \cdot 8$

${}_{11}P_4 = 7920$

There are 7920 ways the coach can select the starting five players, if the tallest student must start at the centre position.

c) ${}_{10}P_3 = \frac{10!}{(10-3)!}$

${}_{10}P_3 = \frac{10!}{7!}$

${}_{10}P_3 = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!}$

${}_{10}P_3 = 10 \cdot 9 \cdot 8$

${}_{10}P_3 = 720$

Multiply by 2, since Sandy and Natasha can play the guard positions in either order. $720 \times 2 = 1440$

There are 1440 ways in which the coach can select the starting five players, if Sandy and Natasha must play the two guard positions.

11. a) $n \geq 0$ and
$n-1 \geq 0$
$n \geq 1$
Therefore, the expression is defined for $n \geq 1$, where $n \in I$.
b) $n+2 \geq 0$
$n \geq -2$
Therefore, the expression is defined for $n \geq -2$, where $n \in I$.
c) $n+1 \geq 0$ AND $n \geq 0$
$n \geq -1$
Therefore, the expression is defined for $n \geq 0$, where $n \in I$.
d) $n+5 \geq 0$ AND $n+3 \geq 0$
$n \geq -5$ $\quad$ $n \geq -3$
Therefore, the expression is defined for $n \geq -3$, where $n \in I$.

12. a) ${}_6P_4 = \dfrac{6!}{(6-4)!}$

$${}_6P_4 = \frac{6!}{2!}$$
$${}_6P_4 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!}$$
$${}_6P_4 = 6 \cdot 5 \cdot 4 \cdot 3$$
$${}_6P_4 = 360$$

There are 360 ways to draw the four marbles if you do not replace the marble each time.
b) Let L represent the number of ways:
$L = 6 \cdot 6 \cdot 6 \cdot 6$
$L = 6^4$
$L = 1296$
There are 1296 ways to draw the four marbles if you replace the marble each time.
c) e.g., Yes; if you replace the marble, there are more possibilities for the next draw.

13. a) ${}_{20}P_5 = \dfrac{20!}{(20-5)!}$

$${}_{20}P_5 = \frac{20!}{15!}$$
$${}_{20}P_5 = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!}{15!}$$
$${}_{20}P_5 = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$$
$${}_{20}P_5 = 1860480$$

There are 1 860 480 different ways to award the scholarships.
b) Let L represent the number of ways:
$L = 20 \cdot 20 \cdot 20 \cdot 20 \cdot 20$
$L = 20^5$
$L = 3\,200\,000$
There are 3 200 000 different ways to award the scholarships.

14. a) ${}_{10}P_4 = \dfrac{10!}{(10-4)!}$

$${}_{10}P_4 = \frac{10!}{6!}$$
$${}_{10}P_4 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!}$$
$${}_{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7$$
$${}_{10}P_4 = 5040$$

There are 5040 different phone numbers possible.
b) Subtract the total possible numbers by the answer to part a).
104 = 10 000
10 000 – 5040 = 4960
There are 4960 different phone numbers.

15. a) I need to solve $\dfrac{n!}{(n-2)!} = 20$.

$n \geq 0$ AND $n-2 \geq 0$
$n \geq 2$
Therefore, $\dfrac{n!}{(n-2)!} = 20$ is defined for $n \geq 2$, where $n \in I$.

$$\frac{n!}{(n-2)!} = 20$$
$$\frac{(n)(n-1)(n-2)(n-3)\ldots(3)(2)(1)}{(n-2)(n-3)\ldots(3)(2)(1)} = 20$$
$$\frac{(n)(n-1)(n-2)!}{(n-2)!} = 20$$
$$(n)(n-1) = 20$$
$$n^2 - n = 20$$
$$n^2 - n - 20 = 0$$
$$(n+4)(n-5) = 0$$

$n+4=0$ or $n-5=0$
$n=-4$ $\quad$ $n=5$
The root $n = -4$ is not a solution to $n \geq 2$.
Check $n = 5$

LS	RS
${}_5P_2$	20
$\dfrac{5!}{(5-2)!}$	
$\dfrac{5!}{3!}$	
$\dfrac{5 \cdot 4 \cdot 3!}{3!}$	
$5 \cdot 4$	
20	

There is one solution, $n = 5$.

b) I need to solve $\frac{(n+1)!}{(n+1-2)!}=72$.

$n+1\geq 0$ AND $n+1-2\geq 0$

$n\geq -1$ $\quad$ $n-1\geq 0$

$n\geq 1$

Therefore, $\frac{(n+1)!}{(n+1-2)!}=72$ is defined for $n \geq 1$, where $n \in I$.

$$\frac{(n+1)!}{(n+1-2)!}=72$$

$$\frac{(n+1)!}{(n-1)!}=72$$

$$\frac{(n+1)(n)(n-1)(n-2)...(3)(2)(1)}{(n-1)(n-2)...(3)(2)(1)}=72$$

$$\frac{(n+1)(n)(n-1)!}{(n-1)!}=72$$

$$(n+1)(n)=72$$

$$n^2+n=72$$

$$n^2+n-72=0$$

$$(n+9)(n-8)=0$$

$n+9=0$ or $n-8=0$

$n=-9$ $\quad$ $n=8$

The root $n = -9$ is not a solution to $n \geq 1$. Check $n = 8$.

LS	RS
${}_{8+1}P_2$	72
${}_9P_2$	
$\frac{9!}{(9-2)!}$	
$\frac{9!}{7!}$	
$\frac{9\cdot 8\cdot 7!}{7!}$	
$9\cdot 8$	
72	

There is one solution, $n = 8$.

16. a) The equation I need to solve is $\frac{6!}{(6-r)!}=30$.

$6-r\geq 0$

$r\leq 6$

Therefore, $\frac{6!}{(6-r)!}=30$ is defined for $0 \leq r \leq 6$, where $r \in I$.

$$\frac{6!}{(6-r)!}=30$$

$$\frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(6-r)!}=30$$

$$\frac{720}{(6-r)!}=30$$

$$(6-r)!=\frac{720}{30}$$

$$(6-r)!=24$$

$$6-r=4$$

$$-r=-2$$

$$r=2$$

Check $r = 2$

LS	RS
${}_6P_2$	30
$\frac{6!}{(6-2)!}$	
$\frac{6!}{4!}$	
$\frac{6\cdot 5\cdot 4!}{4!}$	
$6\cdot 5$	
30	

There is one solution, $r = 2$.

b) The equation I need to solve is $2\left[\frac{7!}{(7-r)!}\right]=420$.

$7-r\geq 0$

$r\leq 7$

Therefore, $2\left[\frac{7!}{(7-r)!}\right]=420$ is defined for $0 \leq r \leq 7$, where $r \in I$.

$$2\left[\frac{7!}{(7-r)!}\right]=420$$

$$\frac{7!}{(7-r)!}=210$$

$$\frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(7-r)!}=210$$

$$\frac{5040}{(7-r)!}=210$$

$$(7-r)!=\frac{5040}{210}$$

$$(7-r)!=24$$

$$7-r=4$$

$$-r=-3$$

$$r=3$$

Check $r = 3$

LS	RS
$2({}_7P_3)$	420
$2\left[\frac{7!}{(7-3)!}\right]$	
$2\left(\frac{7!}{4!}\right)$	
$2\left(\frac{7\cdot 6\cdot 5\cdot 4!}{4!}\right)$	
$2(7\cdot 6\cdot 5)$	
$2(210)$	
420	

There is one solution, $r = 3$.

17. e.g.,

LS	RS
${}_nP_n$	${}_nP_{n-1}$
$\frac{n!}{(n-n)!}$	$\frac{n!}{[n-(n-1)]!}$
$\frac{n!}{0!}$	$\frac{n!}{[n-n+1]!}$
$\frac{n!}{1}$	$\frac{n!}{1!}$
$n!$	$\frac{n!}{1}$
	$n!$

LS = RS

18. a) e.g., The formulas for both ${}_nP_n$ and ${}_nP_r$ have a numerator of $n!$. However, the formula for ${}_nP_n$ has a denominator of 1 and the formula for ${}_nP_r$ has a denominator of $(n-r)!$.

b) e.g., A group of friends each order a different flavour of ice cream from a shop with 12 flavours. How many possibilities are there if the group is 12 people? If the group is 7 people?

19. a) $n = 52$ and $r = 5$

$${}_{52}P_5 = \frac{52!}{(52-5)!}$$

$${}_{52}P_5 = \frac{52!}{47!}$$

$${}_{52}P_5 = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48\cdot 47!}{47!}$$

$${}_{52}P_5 = 52\cdot 51\cdot 50\cdot 49\cdot 48$$

$${}_{52}P_5 = 311875200$$

There are 311 875 200 possible arrangements.

b) $n = 26$ and $r = 5$

$${}_{26}P_5 = \frac{26!}{(26-5)!}$$

$${}_{26}P_5 = \frac{26!}{21!}$$

$${}_{26}P_5 = \frac{26\cdot 25\cdot 24\cdot 23\cdot 22\cdot 21!}{21!}$$

$${}_{26}P_5 = 26\cdot 25\cdot 24\cdot 23\cdot 22$$

$${}_{26}P_5 = 7893600$$

$$\text{Likelihood} = \frac{7\ 893\ 600}{311\ 875\ 200}\cdot 100\%$$

Likelihood = $0.025...\cdot 100\%$

Likelihood = $2.531...\%$

Therefore, there is about a 2.53% chance that an arrangement contains black cards only.

c) $n = 13$ and $r = 5$

$${}_{13}P_5 = \frac{13!}{(13-5)!}$$

$${}_{13}P_5 = \frac{13!}{8!}$$

$${}_{13}P_5 = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8!}{8!}$$

$${}_{13}P_5 = 13\cdot 12\cdot 11\cdot 10\cdot 9$$

$${}_{13}P_5 = 154440$$

$$\text{Likelihood} = \frac{154440}{311875200}\cdot 100\%$$

Likelihood = $0.000...\cdot 100\%$

Likelihood = $0.049...\%$

Therefore, there is about a 0.05% chance that an arrangement contains hearts only.

20. e.g., $_{n+1}P_2 - {}_nP_1 = \frac{(n+1)!}{(n+1-2)!} - \frac{n!}{(n-1)!}$

$= \frac{(n+1)!}{(n-1)!} - \frac{n!}{(n-1)!}$

$= \frac{(n+1)! - n!}{(n-1)!}$

$= \frac{n+1 \cdot n \cdot n-1 \cdot n-2... - n \cdot n-1 \cdot n-2...}{n-1 \cdot n-2...}$

$= \frac{n \cdot n \cdot -1 \cdot n-2...(n+1-1)}{n-1 \cdot n-2...}$

$= n(n)$

$= n^2$

21. e.g., $_nP_{r+1} = \frac{n!}{[n-(r+1)]!}$

$= \frac{n!}{(n-r-1)!}$

$= \frac{n!}{(n-r-1)!} \cdot \frac{n-r}{n-r}$

$= \frac{(n-r)n!}{(n-r)!}$

$= (n-r)\frac{n!}{(n-r)!}$

$= (n-r)_nP_r$

Math in Action, page 95

a) e.g., January 5, April 23, July 24, and October 15 would be 5, 113, 205, and 288.

b) $365 \cdot 365 \cdot 365 \cdot 365 = 17\,748\,900\,630$

c) i) $\frac{_{365}P_4}{365^4} = 0.983...$ or about 98.4%

ii) $1 - \frac{_{365}P_4}{365^4} = 0.016...$ or about 1.6%

d) e.g.: **i)** $\frac{29}{30} = 0.966...$ or 96.7%

ii) $1 - \frac{29}{30} = 0.033...$ or 3.3%

e) For example, they were close but not the same.

Mid-Chapter Review, page 97

1. The number of subs to choose from, S, is based on the number of buns (b), the number of cold cuts (cc), the number of cheeses (c), the number of toppings (t), and the number of sauces (s):

S = (# of b) · (# of cc) · (# of c) · (# of t) · (# of s)

$S = 3 \cdot 5 \cdot 3 \cdot 12 \cdot 3$

$S = 1620$

So, Mario can choose from 1620 different subs.

2. e.g., You can use one of K, W, and C for the first character, one of the 26 uppercase letters for the second and third characters, and one of the 26 uppercase letters or a blank for the last character. From this, I get the following calculation:

# of station names = $3 \cdot 26 \cdot 26 \cdot 27$

# of station names = 54 756

Therefore, 54 756 station names are possible.

3. Event A: Rolling a 2 OR

Event B: Rolling a 10

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

From the table above, there is one way to roll a sum of 2 with a pair of dice and three ways to roll a sum of 10 with a pair of dice.

$n(A \cup B) = n(A) + n(B)$

$n(A \cup B) = 1 + 3$

$n(A \cup B) = 4$

There are 4 ways that a sum of 2 or a sum of 10 can be rolled with a pair of dice.

4. $10 \cdot 9 \cdot 8 = 720$

There are 720 ways to select 3 horses to come first, second, third in a 10-horse race.

5. a) $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$8! = 40\,320$

b) $6! \cdot 3! = (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (3 \cdot 2 \cdot 1)$

$6! \cdot 3! = (720) \cdot (6)$

$6! \cdot 3! = 4320$

c) $\frac{9!}{6!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$\frac{9!}{6!} = 9 \cdot 8 \cdot 7 \cdot \frac{6}{6} \cdot \frac{5}{5} \cdot \frac{4}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$

$\frac{9!}{6!} = 9 \cdot 8 \cdot 7 \cdot \frac{6!}{6!}$

$\frac{9!}{6!} = 9 \cdot 8 \cdot 7 \cdot 1$

$\frac{9!}{6!} = 504$

d) $\frac{10\cdot 9!}{5\cdot 8!}=\frac{10\cdot(9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)}{5\cdot(8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)}$

$\frac{10\cdot 9!}{5\cdot 8!}=\frac{10}{5}\cdot 9\cdot\frac{8}{8}\cdot\frac{7}{7}\cdot\frac{6}{6}\cdot\frac{5}{5}\cdot\frac{4}{4}\cdot\frac{3}{3}\cdot\frac{2}{2}\cdot\frac{1}{1}$

$\frac{10\cdot 9!}{5\cdot 8!}=\frac{10}{5}\cdot 9\cdot\frac{8!}{8!}$

$\frac{10\cdot 9!}{5\cdot 8!}=\frac{10}{5}\cdot 9\cdot 1$

$\frac{10\cdot 9!}{5\cdot 8!}=2\cdot 9$

$\frac{10\cdot 9!}{5\cdot 8!}=18$

6. There are nine players on the team so there are 9 different positions. Let L represent the number of permutations:

$L = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

$L = 9!$

$L = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

$L = 72\cdot 42\cdot 20\cdot 6$

$L = 362\,880$

There are 362 880 different lineups that can be formed by nine players on a softball team.

7. a) $(n+5)(n+4)!$

$=(n+5)\left[(n+4)(n+3)(n+2)...(3)(2)(1)\right]$

$=(n+5)(n+4)(n+3)(n+2)...(3)(2)(1)$

$=(n+5)!$

b) $\frac{(n+4)!}{(n+2)!}=\frac{(n+4)(n+3)(n+2)(n+1)(n)...(3)(2)(1)}{(n+2)(n+1)(n)...(3)(2)(1)}$

$\frac{(n+4)!}{(n+2)!}=\frac{(n+4)(n+3)(n+2)!}{(n+2)!}$

$\frac{(n+4)!}{(n+2)!}=(n+4)(n+3)$

$\frac{(n+4)!}{(n+2)!}=n^2+4n+3n+12$

$\frac{(n+4)!}{(n+2)!}=n^2+7n+12$

c) $\frac{(n-4)!}{(n-5)!}=\frac{(n-4)(n-5)(n-6)(n-7)...(3)(2)(1)}{(n-5)(n-6)(n-7)...(3)(2)(1)}$

$\frac{(n-4)!}{(n-5)!}=\frac{(n-4)(n-5)!}{(n-5)!}$

$\frac{(n-4)!}{(n-5)!}=n-4$

d) $\frac{(n+2)!}{n!}=\frac{(n+2)(n+1)(n)(n-1)(n-2)...(3)(2)(1)}{(n)(n-1)(n-2)...(3)(2)(1)}$

$\frac{(n+2)!}{n!}=\frac{(n+2)(n+1)(n!)}{n!}$

$\frac{(n+2)!}{n!}=(n+2)(n+1)$

$\frac{(n+2)!}{n!}=n^2+n+2n+2$

$\frac{(n+2)!}{n!}=n^2+3n+2$

8. a) $\frac{n!}{(n-2)!}=72$

$\frac{(n)(n-1)(n-2)(n-3)...(3)(2)(1)}{(n-2)(n-3)...(3)(2)(1)}=72$

$\frac{(n)(n-1)(n-2)!}{(n-2)!}=72$

$(n)(n-1)=72$

$n^2-n=72$

$n^2-n-72=0$

$(n+8)(n-9)=0$

$n+8=0$ or $n-9=0$

$n=-8$ $\quad n=9$

Check $n = -8$

LS	RS
$\frac{(-8)!}{(-8-2)!}$	72
$\frac{(-8)!}{(-10)!}$ is undefined	

Check $n = 9$

LS	RS
$\frac{9!}{(9-2)!}$	72
$\frac{9!}{7!}$	
$\frac{9\cdot 8\cdot 7!}{7!}$	
$9\cdot 8$	
72	

There is one solution, $n = 9$.

b)
$$\frac{(n-1)!}{(n-3)!}=30$$
$$\frac{(n-1)(n-2)(n-3)(n-4)...(3)(2)(1)}{(n-3)(n-4)...(3)(2)(1)}=30$$
$$\frac{(n-1)(n-2)(n-3)!}{(n-3)!}=30$$
$$(n-1)(n-2)=30$$
$$n^2-2n-n+2=30$$
$$n^2-3n+2=30$$
$$n^2-3n-28=0$$
$$(n+4)(n-7)=0$$

$n+4=0$ or $n-7=0$

$n=-4$ $\quad$ $n=7$

Check $n=-4$

LS	RS
$\frac{(-4-1)!}{(-4-3)!}$	30
$\frac{(-5)!}{(-7)!}$ is undefined	

Check $n=7$

LS	RS
$\frac{(7-1)!}{(7-3)!}$	30
$\frac{6!}{4!}$	
$\frac{6\cdot5\cdot4!}{4!}$	
$6\cdot5$	
30	

There is one solution, $n=7$.

9. a) $${}_9P_2=\frac{9!}{(9-2)!}$$
$${}_9P_2=\frac{9!}{7!}$$
$${}_9P_2=\frac{9\cdot8\cdot7!}{7!}$$
$${}_9P_2=9\cdot8$$
$${}_9P_2=72$$

b) $${}_{12}P_8=\frac{12!}{(12-8)!}$$
$${}_{12}P_8=\frac{12!}{4!}$$
$${}_{12}P_8=\frac{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4!}{4!}$$
$${}_{12}P_8=12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5$$
$${}_{12}P_8=19958400$$

c) $${}_5P_5=5!$$
$${}_5P_5=5\cdot4\cdot3\cdot2\cdot1$$
$${}_5P_5=120$$

d) $${}_{12}P_{10}=\frac{12!}{(12-10)!}$$
$${}_{12}P_{10}=\frac{12!}{2!}$$
$${}_{12}P_{10}=\frac{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2!}{2!}$$
$${}_{12}P_{10}=12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3$$
$${}_{12}P_{10}=239500800$$

10. a) a: $n+5\geq0$ AND $n+4\geq0$

$n\geq-5$ $\quad$ $n\geq-4$

The expression is defined for $n\geq-4$, where $n\in I$.

b: $n+4\geq0$ AND $n+2\geq0$

$n\geq-4$ $\quad$ $n\geq-2$

The expression is defined for $n\geq-2$, where $n\in I$.

c: $n-4\geq0$ AND $n-5\geq0$

$n\geq4$ $\quad$ $n\geq5$

The expression is defined for $n\geq5$, where $n\in I$.

d: $n+2\geq0$ AND $n\geq0$

$n\geq-2$

The expression is defined for $n\geq0$, where $n\in I$.

b) a: $n\geq0$ AND $n-2\geq0$

$n\geq2$

The expression is defined for $n\geq2$, where $n\in I$.

b: $n-1\geq0$ AND $n-3\geq0$

$n\geq1$ $\quad$ $n\geq3$

The expression is defined for $n\geq3$, where $n\in I$.

11. $n=20$ and $r=6$

$${}_{20}P_6=\frac{20!}{(20-6)!}$$
$${}_{20}P_6=\frac{20!}{14!}$$
$${}_{20}P_6=\frac{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14!}{14!}$$
$${}_{20}P_6=20\cdot19\cdot18\cdot17\cdot16\cdot15$$
$${}_{20}P_6=27907200$$

Rennie can load his CD player in 27 907 200 different ways.

12. $n = 14$ and $r = 2$

$${}_{14}P_2 = \frac{14!}{(14-2)!}$$

$${}_{14}P_2 = \frac{14!}{12!}$$

$${}_{14}P_2 = \frac{14 \cdot 13 \cdot 12!}{12!}$$

$${}_{14}P_2 = 14 \cdot 13$$

$${}_{14}P_2 = 182$$

There are 182 ways that Manny and 2 other players can line up to receive the championship trophy.

13. Agree. e.g., The number of ways to choose a president and a vice-president from a group of five students is $\frac{5!}{(5-2)!} = 20$. I could also use the Fundamental Counting Principle because there are five choices for president and four choices remaining for vice-president: $5 \cdot 4 = 20$.

Lesson 2.4: Permutations When Objects Are Identical, page 104

1. a) $\frac{7!}{3!2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1) \cdot (2 \cdot 1)}$

$\frac{7!}{3!2!} = \frac{7 \cdot 3 \cdot 2 \cdot 5 \cdot 4 \cdot 3}{(3 \cdot 2 \cdot 1)}$

$\frac{7!}{3!2!} = 7 \cdot 5 \cdot 4 \cdot 3$

$\frac{7!}{3!2!} = 420$

b) $\frac{8!}{2!2!2!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2 \cdot 1 \cdot 2!2 \cdot 1}$

$\frac{8!}{2!2!2!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 3$

$\frac{8!}{2!2!2!} = 5040$

c) $\frac{10!}{4!3!2!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}$

$\frac{10!}{4!3!2!} = 10 \cdot 9 \cdot 7 \cdot 5 \cdot 4$

$\frac{10!}{4!3!2!} = 12600$

d) $\frac{12!}{2!4!5!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$\frac{12!}{2!4!5!} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 7$

$\frac{12!}{2!4!5!} = 83160$

2. Let A represent the arrangement of 6 flags:

$$A = \frac{6!}{2!2!2!}$$

$$A = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1}$$

$$A = 6 \cdot 5 \cdot 3$$

$$A = 90$$

There are 90 different signals that can be made from the 6 flags hung in a vertical line.

3. Let C represent the number of ways:

$$C = \frac{6!}{3!3!}$$

$$C = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1}$$

$$C = 5 \cdot 4$$

$$C = 20$$

There are 20 different ways three coins land as heads and three coins land as tails.

4. Let R represent the number of ways:

$$R = \frac{18!}{10!5!3!}$$

$$R = \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3!}$$

$$R = 17 \cdot 14 \cdot 13 \cdot 11 \cdot 9 \cdot 8$$

$$R = 2\,450\,448$$

There are 2 450 448 ways that this record could have occurred.

5. Let C represent the number of ways:

$$C = \frac{9!}{2!3!4!}$$

$$C = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$C = 9 \cdot 7 \cdot 5 \cdot 4$$

$$C = 1260$$

There are 1260 ways that Norm can distribute 1 cookie to each grandchild.

6. a) Let A represent the number of arrangements:

$A = 5!$

$A = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$A = 120$

There are 120 different arrangements that can be made using all the letters.

b) Let A represent the number of arrangements:

$$A = \frac{7!}{2!}$$

$$A = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!}$$

$$A = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$$

$$A = 2520$$

There are 2520 different arrangements that can be made using all the letters.

c) Let A represent the number of arrangements:

$$A=\frac{8!}{2!}$$

$$A=\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2!}{2!}$$

$$A=8\cdot7\cdot6\cdot5\cdot4\cdot3$$

$$A=20160$$

There are 20 160 different arrangements that can be made using all the letters.

d) Let A represent the number of arrangements:

$$A=\frac{12!}{2!3!}$$

$$A=\frac{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1}$$

$$A=11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$$

$$A=39916800$$

There are 39 916 800 different arrangements that can be made using all the letters.

7. a) Let A represent the number of arrangements:

$$A=\frac{15!}{5!5!5!}$$

$$A=\frac{15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1\cdot5\cdot4\cdot3\cdot2\cdot1\cdot5\cdot4\cdot3\cdot2\cdot1}$$

$$A=14\cdot13\cdot11\cdot9\cdot7\cdot6$$

$$A=756756$$

There are 756 756 different ways he can arrange the books on the shelf.

b) Group the sets of 5 together.

$A = 3 \cdot 2 \cdot 1$

$A = 6$

There are 6 ways he can arrange the books.

8. e.g., A shish kabob skewer has 4 pieces of beef, 2 pieces of green pepper, and 1 piece each of mushroom and onion. How many different combinations are possible?

9. a) Let R represent the number of routes:

$$R=\frac{9!}{5!4!}$$

$$R=\frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1\cdot4\cdot3\cdot2\cdot1}$$

$$R=7\cdot6\cdot4\cdot3$$

$$R=126$$

There are 126 routes travelling from point A to point B if you travel only south or east.

b) Let R represent the number of routes:

$$R=\frac{13!}{7!6!}$$

$$R=\frac{13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

$$R=13\cdot11\cdot4\cdot3$$

$$R=1716$$

There are 1716 routes travelling from point A to point B if you travel only south or east.

10. Let R represent the number of routes:

$$R=\frac{13!}{8!5!}$$

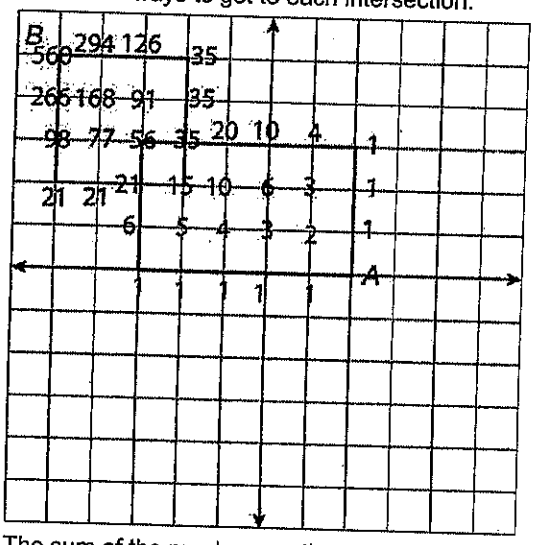
$$R=\frac{13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot5\cdot4\cdot3\cdot2\cdot1}$$

$$R=13\cdot11\cdot9$$

$$R=1287$$

There are 1287 routes travelling from the house of Jess to the house of her friend if she travels only north or west.

11. a) e.g., I drew the following diagram to show the number of ways to get to each intersection.



The sum of the numbers on the top right and bottom left corners of each block is equal to the number of routes to the top left corner of each block. There are 560 different routes from A to B, if you travel only north or west.

b) I need to go north twice and west four times, for a total of 6 moves, to travel the first 2 by 4 block of the route. I need to go north once and west once, for a total of 2 moves, to travel the next 1 by 1 block of the route. I need to go north twice and west twice, for a total of 4 moves, to travel the last 2 by 2 block of the route.

Let R represent the number of routes:

$$R = \frac{6!}{4!2!} \cdot 2! \cdot \frac{4!}{2!2!}$$

$$R = \left(\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}\right) \cdot (2 \cdot 1) \cdot \left(\frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1}\right)$$

$$R = (5 \cdot 3) \cdot (2) \cdot (6)$$

$$R = 180$$

There are 180 different routes from A to B, if you travel only north or west.

12. Let P represent the number of permutations:

$$P = \frac{8!}{5!3!}$$

$$P = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1}$$

$$P = 7 \cdot 4 \cdot 2$$

$$P = 56$$

There are 56 different permutations of answers that the teacher can create.

13. a) Let P represent the number of permutations:

$$P = 7!$$

$$P = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$P = 5040$$

There are 5040 different arrangements possible for the new totem pole.

b) Let P represent the number of permutations:

$$P = \frac{7!}{2!2!}$$

$$P = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1}$$

$$P = 7 \cdot 6 \cdot 5 \cdot 3 \cdot 2 \cdot 1$$

$$P = 1260$$

There are 1260 different arrangements possible for the new totem pole.

14. e.g., ${}_nP_n$ will be too high; it gives the number of arrangements of all n items, but some of the arrangements will be identical because of the a identical items in the group.

15. a) e.g., I am assuming that the coins of the same denomination are considered identical objects. Let A represent the number of arrangements:

$$A = \frac{9!}{4!3!2!}$$

$$A = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 2}$$

$$A = 1260$$

There are 1260 ways the 9 coins can be arranged in a line.

b) e.g., I am assuming that the coins of the same denomination are considered identical objects. Let A represent the number of arrangements:

$$A = \frac{7!}{4!3!}$$

$$A = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 3 \cdot 2}$$

$$A = 35$$

There are 35 ways the 9 coins can be arranged in a line.

16. The number of ways to divide the 8 remaining freezies amongst the other 8 children is what I want. Let P represent the number of permutations:

$$P = \frac{8!}{2!5!}$$

$$P = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$P = 7 \cdot 6 \cdot 4$$

$$P = 168$$

There are 168 ways to distribute the 10 freezies amongst the 10 children.

17. a) Let P represent the number of permutations:

$$P = 1 \cdot \frac{8!}{3!3!2!} \cdot 1$$

$$P = 1 \cdot \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} \cdot 1$$

$$P = 1 \cdot 8 \cdot 7 \cdot 5 \cdot 2 \cdot 1$$

$$P = 560$$

There are 560 permutations possible if you must start with A and end with C.

b) e.g., If you start by putting the I's in the first and second positions, and then in the second and third positions, and so on and so forth up until you put them in the ninth and tenth positions, there are 9 different arrangements of the I's just on their own. The number of different arrangements of all the letters in each of these 9 arrangements is the number of ways to organize the other 8 letters. Since the other 8 letters are always the same, the number of permutations of the letters for each arrangement of the I's is the same. Let P represent the number of permutations:

$$P = 9\left(\frac{8!}{3!3!}\right)$$

$$P = 9\left(\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1}\right)$$

$$P = 9(8 \cdot 7 \cdot 5 \cdot 4)$$

$$P = 9(1120)$$

$$P = 10080$$

There are 10 080 permutations possible if the two I's must be together.

18. e.g., BANDITS has 7 different letters, so the number of permutations is 7! BANANAS also has 7 letters, but there are 3 As and 2 Ns so you must divide 7! by $3! \cdot 2! = 12$.

19. The shortest possible route contains 3 moves diagonally to the right, 3 moves diagonally to the left, and 3 moves down. Let R represent the number of routes;

$$R = \frac{9!}{3!3!3!}$$

$$R = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1}$$

$$R = 8 \cdot 7 \cdot 5 \cdot 3 \cdot 2$$

$$R = 1680$$

There are 1680 routes from the top rear vertex of the cube to the lower front vertex of the cube.

20. a) e.g., This is the same as arranging the 20 players then dividing by 2! ten times because the order of pairs does not matter. Let T represent the number of pairs:

$$T = \frac{20!}{2!^{10}}$$

$$T \doteq 2.375... \times 10^{15}$$

There are about 2.38×10^{15} ways to assign 20 players to 10 double rooms.

b) e.g., This is the same as arranging the 20 players then dividing by 4! five times because the order of pairs does not matter. Let T represent the number of pairs:

$$T = \frac{20!}{4!^{5}}$$

$$T \doteq 3.055... \times 10^{11}$$

There are about 3.06×10^{11} ways to assign 20 players to 5 quadruple rooms.

21. a) e.g., I can make a table to show all of the arrangements that could be made. Position 1 in the table below is the leftmost position, and position 4 is the rightmost position.

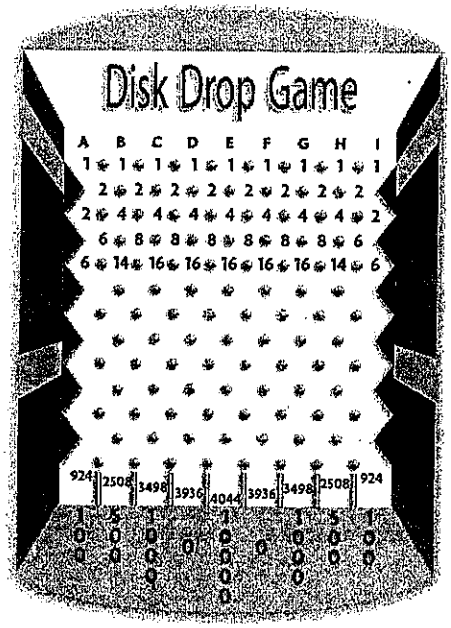
	Position 1	Position 2	Position 3	Position 4
1	R	R	R	W
2	R	R	W	R
3	R	R	W	W
4	R	W	R	R
5	R	W	R	W
6	R	W	W	R
7	R	W	W	W
8	W	R	R	R
9	W	R	R	W
10	W	R	W	R
11	W	R	W	W
12	W	W	R	R
13	W	W	R	W
14	W	W	W	R

From the table, I see that 14 different arrangements might be made.

b) e.g., From the table above, 1 out of the 14 arrangements, from left to right, would be red, white, white, red. Therefore, there is a 1 in 14 chance that the arrangement, from left to right, would be red, white, white, red.

Applying Problem-Solving Strategies, page 108

A. 4044 paths



B. 2(924) + 2(2508) + 2(3498) = 13 860 paths

C. Yes. There are 2(3936), or 7872, paths that lead to no money at all, but 17 904 paths that result in the contestant winning something. The contestant has a better chance of winning something than nothing, so it's a fair game from the contestant's point of view.

Lesson 2.5: Exploring Combinations, page 110

1. a) Let W represent the number of ways:

$W = 3!$

$W = 3 \cdot 2 \cdot 1$

$W = 6$

There are 6 different ways that Brian, Rachelle, and Linh can be chosen for these jobs.

b)

Canned Goods	Dry Goods	Fruits and Vegetables
Brian	Rachelle	Linh
Brian	Linh	Rachelle
Rachelle	Brian	Linh
Rachelle	Linh	Brian
Linh	Rachelle	Brian
Linh	Brian	Rachelle

c) Since all 3 volunteers are being used to help unload the vehicles, there is only one way they can be chosen for this job.
d) Part a) and b) involve permutations and part c) involved combinations. I know because in part a) and b), the order in which the volunteers were selected for the jobs mattered. In part c) the order did not since all the volunteers were being selected to do the same job.

2. e.g., The main difference is that for the permutations, the order of the 4 objects matters, and for the combinations, it does not. For the permutations, you could have multiple arrangements with the same objects since there is more than one way to order a group of four different objects. This is not possible for combinations since you just need one arrangement for each group of 4, regardless of the order.

3. Let C represent the number of dance committees possible:
$C = {}_{10}C_4$
$C = 210$
There are 210 ways that 4 of the members can be chosen to serve on the dance committee.

4. Let C represent the number of combinations:
$C = {}_{12}C_3$
$C = 220$
There are 220 ways 3 of the 12 dogs can be selected to appear.

Lesson 2.6: Combinations, page 118

1. a)

Flavour 1	Flavour 2
vanilla	strawberry
vanilla	chocolate
vanilla	butterscotch
strawberry	vanilla
strawberry	chocolate
strawberry	butterscotch
chocolate	vanilla
chocolate	strawberry
chocolate	butterscotch
butterscotch	vanilla
butterscotch	strawberry
butterscotch	chocolate

b)

Flavour 1	Flavour 2
vanilla	strawberry
vanilla	chocolate
vanilla	butterscotch
strawberry	chocolate
strawberry	butterscotch
chocolate	butterscotch

c) The number of two-flavour permutations is double the number of two-flavour combinations because each two-flavour combination can be written in two different ways.

2. a) Let C represent the number of committees:

$$C = {}_5C_3$$
$$C = \frac{5!}{3!(5-3)!}$$
$$C = \frac{5!}{3!\cdot 2!}$$
$$C = \frac{5\cdot 4\cdot 3!}{3!\cdot 2\cdot 1}$$
$$C = \frac{5\cdot 4}{2\cdot 1}$$
$$C = 5\cdot 2$$
$$C = 10$$

There are 10 possible committees.

b) Let C represent the number of committees:

$$C = {}_5C_2$$
$$C = \frac{5!}{2!(5-2)!}$$
$$C = \frac{5!}{2!\cdot 3!}$$
$$C = \frac{5\cdot 4\cdot 3!}{2\cdot 1\cdot 3!}$$
$$C = \frac{5\cdot 4}{2\cdot 1}$$
$$C = 5\cdot 2$$
$$C = 10$$

There are 10 possible committees.

c) e.g., My answers for parts a) and b) are the same. This occurred because the sum of 2 and 3 is 5.

3. Let T represent the number of possible teams:

$T = {}_{12}C_6$

$T = \frac{12!}{6!(12-6)!}$

$T = \frac{12!}{6!6!}$

$T = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 6!}$

$T = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$T = 2 \cdot 11 \cdot 5 \cdot 3 \cdot 2 \cdot \frac{7}{5}$

$T = 924$

There are 924 ways 6 people can be selected from a group of 12 to form a dodge-ball team.

4. a) ${}_5C_3 = \frac{5!}{3!(5-3)!}$

${}_5C_3 = \frac{5!}{3!2!}$

${}_5C_3 = \frac{5 \cdot 4 \cdot 3!}{3!2 \cdot 1}$

${}_5C_3 = \frac{5 \cdot 4}{2 \cdot 1}$

${}_5C_3 = 5 \cdot 2$

${}_5C_3 = 10$

b) ${}_9C_8 = \frac{9!}{8!(9-8)!}$

${}_9C_8 = \frac{9!}{8!1!}$

${}_9C_8 = \frac{9 \cdot 8!}{8!1}$

${}_9C_8 = \frac{9}{1}$

${}_9C_8 = 9$

c) ${}_6C_4 = \frac{6!}{4!(6-4)!}$

${}_6C_4 = \frac{6!}{4!2!}$

${}_6C_4 = \frac{6 \cdot 5 \cdot 4!}{4!2 \cdot 1}$

${}_6C_4 = \frac{6 \cdot 5}{2 \cdot 1}$

${}_6C_4 = 3 \cdot 5$

${}_6C_4 = 15$

d) ${}_{10}C_0 = \frac{10!}{0!(10-0)!}$

${}_{10}C_0 = \frac{10!}{0!10!}$

${}_{10}C_0 = \frac{1}{1}$

${}_{10}C_0 = 1$

e) ${}_{12}C_6 = \frac{12!}{6!(12-6)!}$

${}_{12}C_6 = \frac{12!}{6!6!}$

${}_{12}C_6 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 6!}$

${}_{12}C_6 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

${}_{12}C_6 = 2 \cdot 11 \cdot 5 \cdot 3 \cdot 2 \cdot \frac{7}{5}$

${}_{12}C_6 = 924$

f) ${}_8C_1 = \frac{8!}{1!(8-1)!}$

${}_8C_1 = \frac{8!}{1!7!}$

${}_8C_1 = \frac{8 \cdot 7!}{1 \cdot 7!}$

${}_8C_1 = \frac{8}{1}$

${}_8C_1 = 8$

5. Let C represent the number of combinations:

$C = {}_{10}C_6$

$C = \frac{10!}{6!(10-6)!}$

$C = \frac{10!}{6!4!}$

$C = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!4 \cdot 3 \cdot 2 \cdot 1}$

$C = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}$

$C = 5 \cdot 3 \cdot 2 \cdot 7$

$C = 210$

There are 210 ways 6 players can be chosen to start a volleyball game from a team of 10.

6. Let C represent the number of combinations:

$C = {}_{55}C_5$

$C = \frac{55!}{5!(55-5)!}$

$C = \frac{55!}{5!50!}$

$C = \frac{55 \cdot 54 \cdot 53 \cdot 52 \cdot 51 \cdot 50!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 50!}$

$C = \frac{55 \cdot 54 \cdot 53 \cdot 52 \cdot 51}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$C = 11 \cdot 27 \cdot 53 \cdot 13 \cdot 17$

$C = 3\ 478\ 761$

There are 3 478 761 different combinations of hip-hop songs you can download for free.

7. Let H represent the number of hands:

$H = {}_{52}C_8$

$H = \frac{52!}{8!(52-8)!}$

$H = \frac{52!}{8!44!}$

$H = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44!}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 44!}$

$H = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$H = 13 \cdot 17 \cdot 10 \cdot 7 \cdot 47 \cdot 45 \cdot 23$

$H = 752538150$

There are 752 538 150 different 8-card hands that can be dealt.

8. a) The problem involves combinations e.g., because it does not state that the order of the starting line matters.

b) Let L represent the number of different lineups, $n = 14$ and $r = 8$ because Connie must be the pitcher of the starting lineup.

$L = {}_nC_r$

$L = {}_{14}C_8$

$L = \frac{14!}{8!(14-8)!}$

$L = \frac{14!}{8!6!}$

$L = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8!}{8!6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$L = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$L = 7 \cdot 13 \cdot 11 \cdot 3$

$L = 3003$

There are 3003 ways that the coach can choose his starting lineup of 9 players, if Connie must be the pitcher.

9. a) Yes, I do agree.

e.g.,

LS	RS
${}_6C_2$	${}_6C_4$
$\frac{6!}{2!(6-2)!}$	$\frac{6!}{4!(6-4)!}$
$\frac{6!}{2!4!}$	$\frac{6!}{4!2!}$
$\frac{6 \cdot 5 \cdot 4!}{2 \cdot 1 \cdot 4!}$	$\frac{6 \cdot 5 \cdot 4!}{4!2 \cdot 1}$
$\frac{6 \cdot 5}{2 \cdot 1}$	$\frac{6 \cdot 5}{2 \cdot 1}$
$3 \cdot 5$	$3 \cdot 5$
15	15

LS = RS

b) e.g., Some other cases with the same relationship as part a) are ${}_8C_1 = {}_8C_7$, ${}_6C_0 = {}_6C_6$, and ${}_{12}C_7 = {}_{12}C_5$. I notice that if you have two combinations with the same n, and the sum of the r's for those combinations is equal to n, then the value of the combinations will be the same.

c) e.g., $\binom{n}{r} = \binom{n}{n-r}$

10.

Let T represent the number of combinations for the teachers:

$T = {}_5C_2$

$T = \frac{5!}{2!(5-2)!}$

$T = \frac{5!}{2!3!}$

$T = \frac{5 \cdot 4 \cdot 3!}{2 \cdot 1 \cdot 3!}$

$T = \frac{5 \cdot 4}{2 \cdot 1}$

$T = 5 \cdot 2$

$T = 10$

Let S represent the number of combinations for the students:

$S = {}_8C_3$

$S = \frac{8!}{3!(8-3)!}$

$S = \frac{8!}{3!5!}$

$S = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2 \cdot 1 \cdot 5!}$

$S = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$

$S = 4 \cdot 2 \cdot 7$

$S = 56$

Let G represent the number of graduation committees:

$G = S \cdot T$

$G = 56 \cdot 10$

$G = 560$

There are 560 graduation committees that the principal has to choose from.

11. a) Let C represent the number of committees:

$C = {}_{10}C_5$

$C = \frac{10!}{5!(10-5)!}$

$C = \frac{10!}{5!5!}$

$C = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5!}$

$C = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$C = 2 \cdot 3 \cdot 2 \cdot 7 \cdot 3$

$C = 252$

There are 252 committees that can be formed if there are no conditions.

b)

Let W represent the number of combinations for the women:

$W = {}_6C_3$

$W = \frac{6!}{3!(6-3)!}$

$W = \frac{6!}{3!3!}$

$W = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3 \cdot 2 \cdot 1 \cdot 3!}$

$W = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$

$W = 2 \cdot 5 \cdot 2$

$W = 20$

Let M represent the numbe of combinations for the men

$M = {}_4C_2$

$M = \frac{4!}{2!(4-2)!}$

$M = \frac{4!}{2!2!}$

$M = \frac{4 \cdot 3 \cdot 2!}{2 \cdot 1 \cdot 2!}$

$M = \frac{4 \cdot 3}{2 \cdot 1}$

$M = 2 \cdot 3$

$M = 6$

Let C represent the number of committees:
$C = W \cdot M$
$C = 20 \cdot 6$
$C = 120$
There are 120 committees that can be formed if there must be exactly 3 women.
c) Let C represent the number of committees:
$C = {}_6C_1$
$C = \frac{6!}{1!(6-1)!}$
$C = \frac{6!}{1! \cdot 5!}$
$C = \frac{6 \cdot 5!}{1 \cdot 5!}$
$C = \frac{6}{1}$
$C = 6$
There are 6 committees that can be formed if there must be exactly 4 men.

d) Let C represent the number of committees:
$C = {}_6C_5$
$C = \frac{6!}{5!(6-5)!}$
$C = \frac{6!}{5! \cdot 1!}$
$C = \frac{6 \cdot 5!}{5! \cdot 1}$
$C = \frac{6}{1}$
$C = 6$
There are 6 committees that can be formed if there can be no men.
e) e.g., **Case 1:** 3 men and 2 women
${}_4C_3 \cdot {}_6C_2 = \frac{4!}{3! \cdot 1!} \cdot \frac{6!}{2! \cdot 4!}$
${}_4C_3 \cdot {}_6C_2 = 60$

Case 2: 4 men and 1 woman
${}_4C_4 \cdot {}_6C_1 = \frac{4!}{4! \cdot 0!} \cdot \frac{6!}{1! \cdot 5!}$
${}_4C_4 \cdot {}_6C_1 = 1 \cdot 6$
${}_4C_4 \cdot {}_6C_1 = 6$
Number of committees = 60 + 6
Number of committees = 66
66 5-person committees can be formed if there must be at least 3 men.

12. e.g., Let's say I want to assign students to the room with 5 beds first. Let A represent the number of ways to assign the 12 students to the 5 beds:
$A = {}_{12}C_5$
$A = \frac{12!}{5!(12-5)!}$
$A = \frac{12!}{5! \cdot 7!}$
$A = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 7!}$
$A = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$
$A = 3 \cdot 11 \cdot 2 \cdot 3 \cdot 4$
$A = 792$
Now there are 12 – 5 or 7 students left to assign. Let's assign students to the room with 4 beds next. Let B represent the number of ways to assign the 7 students to the 4 beds:
$B = {}_7C_4$
$B = \frac{7!}{4!(7-4)!}$
$B = \frac{7!}{4! \cdot 3!}$
$B = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3 \cdot 2 \cdot 1}$
$B = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}$
$B = 7 \cdot 5$
$B = 35$
Now there are 7 – 4 or 3 students left to assign to the room with 3 beds. Since all of these students will be assigned to the room, there is only one combination for them. Let C now represent the number of different assignments:
$C = A \cdot B \cdot 1$
$C = 792 \cdot 35$
$C = 27\,720$
There are 27 720 ways the 12 students can be assigned to these rooms.

13. a) i) 5 objects, 3 in each combination
ii) 10 objects, 2 in each combination
iii) 5 objects, 3 in each combination
b) e.g., **i)** How many ways can you choose 3 coins from a bag containing a penny, a nickel, a dime, a quarter, and a loonie?

14. a) i) ${}_0C_0 = \dfrac{0!}{0!(0-0)!}$

$${}_0C_0 = \frac{0!}{0!\cdot 0!}$$

$${}_0C_0 = \frac{1}{1}$$

$${}_0C_0 = 1$$

ii) ${}_1C_0 = \dfrac{1!}{0!(1-0)!}$

$${}_1C_0 = \frac{1!}{0!\cdot 1!}$$

$${}_1C_0 = \frac{1}{1}$$

$${}_1C_0 = 1$$

$${}_1C_1 = \frac{1!}{1!(1-1)!}$$

$${}_1C_1 = \frac{1!}{1!\cdot 0!}$$

$${}_1C_1 = \frac{1}{1}$$

$${}_1C_1 = 1$$

${}_1C_0, {}_1C_1 = 1, 1$

iii) ${}_2C_0 = \dfrac{2!}{0!(2-0)!}$

$${}_2C_0 = \frac{2!}{0!\cdot 2!}$$

$${}_2C_0 = \frac{1}{1}$$

$${}_2C_0 = 1$$

${}_2C_1 = \dfrac{2!}{1!(2-1)!}$

$${}_2C_1 = \frac{2!}{1!\cdot 1!}$$

$${}_2C_1 = \frac{2\cdot 1!}{1\cdot 1!}$$

$${}_2C_1 = \frac{2}{1}$$

$${}_2C_1 = 2$$

$${}_2C_2 = \frac{2!}{2!(2-2)!}$$

$${}_2C_2 = \frac{2!}{2!\cdot 0!}$$

$${}_2C_2 = \frac{1}{1}$$

$${}_2C_2 = 1$$

${}_2C_0, {}_2C_1, {}_2C_2 = 1, 2, 1$

iv) ${}_3C_0 = \dfrac{3!}{0!(3-0)!}$

$${}_3C_0 = \frac{3!}{0!\cdot 3!}$$

$${}_3C_0 = \frac{1}{1}$$

$${}_3C_0 = 1$$

${}_3C_1 = \dfrac{3!}{1!(3-1)!}$

$${}_3C_1 = \frac{3!}{1!\cdot 2!}$$

$${}_3C_1 = \frac{3\cdot 2!}{1\cdot 2!}$$

$${}_3C_1 = \frac{3}{1}$$

$${}_3C_1 = 3$$

${}_3C_2 = \dfrac{3!}{2!(3-2)!}$

$${}_3C_2 = \frac{3!}{2!\cdot 1!}$$

$${}_3C_2 = \frac{3\cdot 2!}{2!\cdot 1}$$

$${}_3C_2 = \frac{3}{1}$$

$${}_3C_2 = 3$$

${}_3C_3 = \dfrac{3!}{3!(3-3)!}$

$${}_3C_3 = \frac{3!}{3!\cdot 0!}$$

$${}_3C_3 = \frac{1}{1}$$

$${}_3C_3 = 1$$

${}_3C_0, {}_3C_1, {}_3C_2, {}_3C_3 = 1, 3, 3, 1$

v) ${}_4C_0 = \dfrac{4!}{0!(4-0)!}$

$${}_4C_0 = \frac{4!}{0!\cdot 4!}$$

$${}_4C_0 = \frac{1}{1}$$

$${}_4C_0 = 1$$

${}_4C_1 = \dfrac{4!}{1!(4-1)!}$

$${}_4C_1 = \frac{4!}{1!\cdot 3!}$$

$${}_4C_1 = \frac{4\cdot 3!}{1\cdot 3!}$$

$${}_4C_1 = \frac{4}{1}$$

$${}_4C_1 = 4$$

$${}_4C_2 = \frac{4!}{2!(4-2)!}$$

$${}_4C_2 = \frac{4!}{2!\cdot 2!}$$

$${}_4C_2 = \frac{4\cdot 3\cdot 2!}{2\cdot 1\cdot 2!}$$

$${}_4C_2 = \frac{4\cdot 3}{2\cdot 1}$$

$${}_4C_2 = 2\cdot 3$$

$${}_4C_2 = 6$$

${}_4C_3 = \dfrac{4!}{3!(4-3)!}$

$${}_4C_3 = \frac{4!}{3!\cdot 1!}$$

$${}_4C_3 = \frac{4\cdot 3!}{3!\cdot 1}$$

$${}_4C_3 = \frac{4}{1}$$

$${}_4C_3 = 4$$

$${}_4C_4 = \frac{4!}{4!(4-4)!}$$

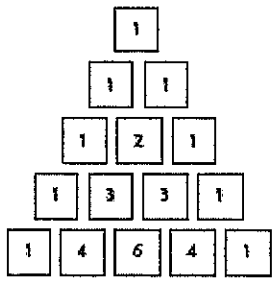
$${}_4C_4 = \frac{4!}{4!\cdot 0!}$$

$${}_4C_4 = \frac{1}{1}$$

$${}_4C_4 = 1$$

${}_4C_0, {}_4C_1, {}_4C_2, {}_4C_3, {}_4C_3 = 1, 4, 6, 4, 1$

b)



c) e.g., The numbers on the left and right sides are all 1s; every other number is the sum of the two numbers above it.
d) sixth row: 1, 5, 10, 10, 5,1
seventh row: 1, 6, 15, 20, 15, 6, 1
e) e.g., The number in each square of Pascal's Triangle is equal to the number of pathways to it from the top square.

15. a) The equation I need to solve is $\frac{n!}{2!(n-2)!}=15$.

$n \geq 0$ AND $n-2 \geq 0$

$n \geq 2$

$\frac{n!}{2!(n-2)!}=15$ is defined for $n \geq 2$, where $n \in$ N.

$$\frac{n!}{2!(n-2)!}=15$$
$$\frac{n!}{(n-2)!}=15 \cdot 2!$$
$$\frac{n!}{(n-2)!}=15(2)$$
$$\frac{n!}{(n-2)!}=30$$
$$\frac{n(n-1)(n-2)(n-3)...(3)(2)(1)}{(n-2)(n-3)...(3)(2)(1)}=30$$
$$n(n-1)=30$$
$$n^2-n=30$$
$$n^2-n-30=0$$
$$(n+5)(n-6)=0$$

$n+5=0$ OR $n-6=0$

$n=-5$ $\quad$ $n=6$

Based on the restrictions, $n = -5$ cannot be a solution. Therefore, $n = 6$ is the solution to the equation.

b) The equation I need to solve is $\frac{n!}{4!(n-4)!}=35$.

$n \geq 0$ AND $n-4 \geq 0$

$n \geq 4$

$\frac{n!}{4!(n-4)!}=35$ is defined for $n \geq 4$, where $n \in$ N.

$$\frac{n!}{4!(n-4)!}=35$$
$$\frac{n!}{(n-4)!}=35 \cdot 4!$$
$$\frac{n!}{(n-4)!}=840$$
$$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)...(3)(2)(1)}{(n-4)(n-5)...(3)(2)(1)}=840$$
$$n(n-1)(n-2)(n-3)=840$$
$$(n^2-n)(n^2-5n+6)=840$$
$$n^4-5n^3+6n^2-n^3+5n^2-6n=840$$
$$n^4-6n^3+11n^2-6n-840=0$$

Write out all of the factors of –840: ±1, ±2, ±3, ±4, ±5, ±6, ±7, ±8, ±10, ±12, ±14, ±15, ±20, ±21, ±24, ±28, ±30, ±35, ±40, ±42, ±56, ±60, ±70, ±84, ±105, ±120, ±140, ±168, ±210, ±280, ±420, ±840. These are all the possible roots of the equation. Substitute them into the equation and if the equation goes to 0, then I have a root of the equation. By trial and error, the roots are –4 and 7. The other roots are not real. Since –4 is out of the domain, the only real solution is $n = 7$.

c) The equation I need to solve is

$$4\left[\frac{n!}{2!(n-2)!}\right]=\frac{(n+2)!}{3!(n+2-3)!}.$$

$n \geq 0$ AND $n-2 \geq 0$ AND $n+2 \geq 0$

$n \geq 2$ $\quad$ $n \geq -2$

AND $n+2-3 \geq 0$

$n-1 \geq 0$

$n \geq 1$

$4\left[\frac{n!}{2!(n-2)!}\right]=\frac{(n+2)!}{3!(n+2-3)!}$ is defined for $n \geq 2$, where $n \in$ N.

$$4\left[\frac{n!}{2!(n-2)!}\right]=\frac{(n+2)!}{3!(n+2-3)!}$$

$$4\left[\frac{n!}{2(n-2)!}\right]=\frac{(n+2)!}{6(n+2-3)!}$$

$$\frac{2n!}{(n-2)!}=\frac{(n+2)!}{6(n+2-3)!}$$

$$\frac{2n!}{(n-2)!}=\frac{(n+2)!}{6(n-1)!}$$

$$2n(n-1)=\frac{(n+2)(n+1)(n)}{6}$$

$$12n(n-1)=(n+2)(n+1)(n)$$

$$12(n-1)-(n+2)(n+1)=0$$

$$12n-12-(n^2+n+2n+2)=0$$

$$12n-12-n^2-n-2n-2=0$$

$$-n^2+9n-14=0$$

$$-(n-2)(n-7)=0$$

$n-2=0$ or $n-7=0$

$n=2$ $\quad$ $n=7$

Both the roots are within the domain, so there are two solutions, $n = 2$ and $n = 7$.

d) The equation I need to solve is $\frac{6!}{r!(6-r)!}=15$.

$r \geq 0$ AND $6-r \geq 0$

$r \leq 6$

$\frac{6!}{r!(6-r)!}=15$ is defined for $0 \leq r \leq 6$, where $r \in I$.

$$\frac{6!}{r!(6-r)!}=15$$

$$r!(6-r)!=\frac{6!}{15}$$

$$r!(6-r)!=\frac{720}{15}$$

$$r!(6-r)!=48$$

By substituting each of the integers r for $0 \leq r \leq 6$, I get $r = 2$ or $r = 4$.

16. a) 1, e.g., the player can only win if the six numbers they choose are the same and in the same order as the six numbers drawn.

b) ${}_{66}C_6=\frac{66!}{6!(66-6)!}$

${}_{66}C_6=\frac{66!}{6!60!}$

${}_{66}C_6=\frac{66\cdot65\cdot64\cdot63\cdot62\cdot61\cdot60!}{6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot60!}$

${}_{66}C_6=\frac{66\cdot65\cdot64\cdot63\cdot62\cdot61}{6\cdot5\cdot4\cdot3\cdot2\cdot1}$

${}_{66}C_6=11\cdot13\cdot16\cdot21\cdot31\cdot61$

${}_{66}C_6=90858768$

There are 90 858 768 different ways the player can win.

c) e.g., No. Even if everyone in the city plays, it is very unlikely that anyone will win since each player only has a 1 in 90 858 767 chance of winning.

17. e.g., The number of sides in a polygon is equal to the number of vertices, the number of vertices = n. From each vertex, v, a diagonal is formed by a line segment intersecting a vertex that is not directly beside v. Thus, the number of vertices that will make a diagonal with v in an n-sided polygon is $n-2$. Trying $r = 2$ in ${}_nC_r$ with the values from the polygons on the side of the textbook page, there is a pattern: (d = number of diagonals)

n	r	${}_nC_2$	d
3	2	3	0
4	2	6	2
5	2	10	5
6	2	15	9

Notice that $n + d = {}_nC_2$.

Rearranging, $d = {}_nC_2 - n$. Thus, the number of diagonals for an n-sided polygon can be determined using ${}_nC_2 - n$.

18. a) Case 1: 2 boys and 3 girls

${}_7C_2 \cdot {}_{13}C_3=\frac{7!}{2!5!}\cdot\frac{13!}{3!10!}$

${}_7C_2 \cdot {}_{13}C_3=21\cdot286$

${}_7C_2 \cdot {}_{13}C_3=6006$

Case 2: 3 boys and 2 girls

${}_7C_3 \cdot {}_{13}C_2=\frac{7!}{3!4!}\cdot\frac{13!}{2!11!}$

${}_7C_3 \cdot {}_{13}C_2=35\cdot78$

${}_7C_3 \cdot {}_{13}C_2=2730$

Case 3: 4 boys and 1 girl

${}_7C_4 \cdot {}_{13}C_1=\frac{7!}{4!3!}\cdot\frac{13!}{1!12!}$

${}_7C_4 \cdot {}_{13}C_1=35\cdot13$

${}_7C_4 \cdot {}_{13}C_1=455$

Case 4: 5 boys and 0 girls

$${}_7C_5 \cdot {}_{13}C_0 = \frac{7!}{5!2!} \cdot \frac{13!}{0!13!}$$

$${}_7C_5 \cdot {}_{13}C_0 = 21 \cdot 1$$

$${}_7C_5 \cdot {}_{13}C_0 = 21$$

Number of groups = 6006 + 2730 + 455 + 21
Number of groups = 9212
There are 9212 different groups of 5 students with at least 2 boys to choose from.

b) Number of groups with no conditions:

$${}_{20}C_5 = \frac{20!}{5! \cdot 15!}$$

Case 1: 1 boy and 4 girls

$${}_7C_1 \cdot {}_{13}C_4 = \frac{7!}{1!6!} \cdot \frac{13!}{4!9!}$$

$${}_7C_1 \cdot {}_{13}C_4 = 5005$$

Case 2: 0 boys and 5 girls

$${}_7C_0 \cdot {}_{13}C_5 = \frac{7!}{0!7!} \cdot \frac{13!}{5!8!}$$

$${}_7C_0 \cdot {}_{13}C_5 = 1287$$

Number of groups with at least two boys:
${}_{20}C_5 - {}_7C_1 \cdot {}_{13}C_4 - {}_7C_0 \cdot {}_{13}C_5 = 9212$
There are 9212 different groups of 5 students with at least 2 boys to choose from.

c) e.g., I prefer indirect reasoning because fewer calculations are needed.

19. a) e.g., Combinations and permutations both involve choosing objects from a group. For permutations, order matters. For combinations, order does not matter. For example, *abc* and *bac* are different permutations, but the same combination.

b) Divide ${}_nP_r$ by $r!$ to get ${}_nC_r$. For example, ${}_6C_4 = 15$ and ${}_6P_4 = 360$; $\frac{360}{4!} = 15$

20. First, determine the total number of outcomes possible. I'll assume that once a song is selected, it cannot be selected again. The number of outcomes, O, is:

$$O = {}_{71}C_5$$

$$O = \frac{71!}{5!66!}$$

$$O = 13019909$$

a) Number of times the event could occur:

$${}_{26}C_5 = \frac{26!}{5!21!}$$

$${}_{26}C_5 = 65780$$

Probability (P):

$$P = \frac{65780}{13019909} \times 100\%$$

$$P = 0.505...\%$$

There is about a 0.51% chance that the five songs will be from CD 2 and CD 4.

b) Number of times the event could occur:

$$12 \cdot 14 \cdot 15 \cdot 12 \cdot 18 = 544320$$

Probability (P):

$$P = \frac{544320}{13019909} \times 100\%$$

$$P = 4.180...\%$$

There is about a 4.18% chance that one of the five songs will be from each CD.

c) There is only one time where your favourite song from each of the 5 CDs will be played.
Probability (P):

$$P = \frac{1}{13019909} \times 100\%$$

$$P = 0.000008\%$$

There is about a 0.000008% or 1 in 13 019 909 chance that your favourite song from each of the 5 CDs will be played.

21. ${}_nC_3 + {}_nC_2 + {}_nC_1$

$$= \frac{n!}{3!(n-3)!} + \frac{n!}{2!(n-2)!} + \frac{n!}{1!(n-1)!}$$

$$= \frac{n!}{6(n-3)!} + \frac{n!}{2(n-2)!} + \frac{n!}{(n-1)!}$$

$$= \frac{n!}{6(n-3)!} + \frac{3n(n-1)(n-3)!}{6(n-3)!} + \frac{6n(n-3)!}{6(n-3)!}$$

$$= \frac{n! + 3n(n-1)(n-3)! + 6n(n-3)!}{6(n-3)!}$$

$$= \frac{(n-3)!\left[n(n-1)(n-2) + 3n(n-1) + 6n\right]}{6(n-3)!}$$

$$= \frac{n(n-1)(n-2) + 3n(n-1) + 6n}{6}$$

$$= \frac{n(n^2 - 2n - n + 2 + 3n - 3 + 6)}{6}$$

$$= \frac{n(n^2 + 5)}{6}$$

$$= \frac{n^3 + 5n}{6}$$

22. e.g.,

LS	RS
${}_{n+1}C_r$	${}_nC_r + {}_nC_{r-1}$
$\frac{(n+1)!}{r!(n+1-r)!}$	$\frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)![n-(r-1)]!}$
	$\frac{[n-(r-1)]n!}{r![n-(r-1)]!} + \frac{r(n!)}{r![n-(r-1)]!}$
	$\frac{(n+1-r)n! + r(n!)}{r!(n+1-r)!}$
	$\frac{n!(n+1-r+r)}{r!(n+1-r)!}$
	$\frac{n!(n+1)}{r!(n+1-r)!}$
	$\frac{(n+1)!}{r!(n+1-r)!}$
	LS = RS

Therefore, ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$.

Lesson 2.7: Solving Counting Problems, page 126

1. a) This situation involves combinations because the order of the 3 toppings on the pizza does not matter.
b) This situation involves permutations because the three spots for the candidates who are selected are all different so order matters.
c) This situation involves permutations because for a group of 3 numbers, there are different ways to roll those three numbers because of the different colours of the dice.
d) This situation involves combinations because the 5 children who are selected are all in the same position. No information is stated in the question about positions the children may play, so I can only assume that they are not playing in specific positions.

2. e.g., Situation A involves combinations and situation B involves permutations. For situation A, order does not matter since the 3 people who are selected will all be considered equals. For situation B, this is not the case. Each of the 3 people who are selected will have a different position with a different amount of power and different roles.

3. a) ${}_3C_3 = \frac{3!}{3!\cdot 0!}$

${}_3C_3 = 1$

There is 1 way that Maddy can bid on 3 items if she bids on only her 3 favourite items.

b) ${}_8C_3 = \frac{8!}{3!\cdot 5!}$

${}_8C_3 = 56$

There are 56 ways that Maddy can bid on 3 items if she bids on any 3 of the 8 items.

4. $\left({}_{13}C_1\right)^4 = \left(\frac{13!}{1!\cdot 12!}\right)^4$

$\left({}_{13}C_1\right)^4 = (13)^4$

$\left({}_{13}C_1\right)^4 = 28561$

There are 28 561 different four-card hands with one card from each suit.

5. a) ${}_{200}P_5 = \frac{200!}{195!}$

${}_{200}P_5 = \frac{200\cdot 199\cdot 198\cdot 197\cdot 196\cdot 195!}{195!}$

${}_{200}P_5 = 200\cdot 199\cdot 198\cdot 197\cdot 196$

${}_{200}P_5 = 304\ 278\ 004\ 800$

There are 304 278 004 800 ways that the top five cash prizes can be awarded if each ticket is not replaced when drawn.
b) $(200)^5 = 320\ 000\ 000\ 000$
There are 320 000 000 000 ways that the top five cash prizes can be awarded if each ticket is replaced when drawn.

6. ${}_2C_1 \cdot {}_6C_2 \cdot {}_4C_2$

$= \left(\frac{2!}{1!(2-1)!}\right)\cdot\left(\frac{6!}{2!(6-2)!}\right)\cdot\left(\frac{4!}{2!(4-2)!}\right)$

$= 2\cdot\left(\frac{6\cdot 5\cdot 4!}{2!\cdot 4!}\right)\cdot\left(\frac{4\cdot 3\cdot 2!}{2!\cdot 2!}\right)$

$= 2\cdot\left(\frac{6\cdot 5}{2}\right)\cdot\left(\frac{4\cdot 3}{2}\right)$

$= 2\cdot 15\cdot 6$

$= 180$

There are 180 ways that the 5 starting positions on the basketball team can be filled.

7. $\frac{10!}{2!\cdot 2!\cdot 2!\cdot 2!\cdot 2!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 2\cdot 2\cdot 2\cdot 2}$

$\frac{10!}{2!\cdot 2!\cdot 2!\cdot 2!\cdot 2!} = 10\cdot 9\cdot 7\cdot 6\cdot 5\cdot 3\cdot 2\cdot 1$

$\frac{10!}{2!\cdot 2!\cdot 2!\cdot 2!\cdot 2!} = 113400$

There are 113 400 ways that the five different pairs of identical teddy bears can be arranged.

8. Case 1: 3 flags are used

$${}_5P_3 = \frac{5!}{(5-3)!}$$

$${}_5P_3 = \frac{5!}{2!}$$

$${}_5P_3 = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1}$$

$${}_5P_3 = 5 \cdot 4 \cdot 3$$

$${}_5P_3 = 60$$

Case 2: 4 flags are used

$${}_5P_4 = \frac{5!}{(5-4)!}$$

$${}_5P_4 = \frac{5!}{1!}$$

$${}_5P_4 = 5!$$

$${}_5P_4 = 120$$

Case 3: 5 flags are used

${}_5P_5 = 5!$

${}_5P_5 = 120$

Let S represent the number of different signals that can be sent using at least three of the flags:

$S = 60 + 120 + 120$

$S = 300$

There are 300 different signals that can be sent using at least three of the flags.

9. e.g., First make a table to show the number of ways the two cabin cruisers can be arranged next to each other.

	CC 1	CC 2
Arrangement 1	1	2
Arrangement 2	2	3
Arrangement 3	3	4
Arrangement 4	4	5
Arrangement 5	5	6
Arrangement 6	2	1
Arrangement 7	3	2
Arrangement 8	4	3
Arrangement 9	5	4
Arrangement 10	6	5

For each of these arrangements, the number of ways the six boats can dock is the number of ways that the other four boats can dock. Let D represent the total number of ways that the boats can dock:

$D = 10 \cdot 4!$

$D = 10 \cdot 24$

$D = 240$

There are 240 ways that the six boats can dock.

10. e.g., Each row of seats is different, and within a row, the seats are assumed to be different. Therefore, there are 10 different people being seated in 10 different spots. Let A represent the number of seating arrangements:

$A = 10!$

$A = 3\,628\,800$

There are 3 628 800 ways that the 10 players can sit in the van.

11. a) $\frac{5!}{2!} = 60$

There are 60 different arrangements that are possible for the letters if there are no conditions.

b) $3! = 6$

There are 6 different arrangements that are possible for the letters if each arrangement must start and end with an N.

12. e.g., When there is an even amount of numbers, half of them will be odd. In this case there are 100 possible numbers that each number can be.

Therefore, $\frac{100}{2}$, or 50 of them are odd. Since I want each number to only be 1 of these 50 odd numbers, the number of sequences S is:

$S = 50 \cdot 50 \cdot 50$

$S = 125\,000$

There are 125 000 completely odd sequences.

13. $\frac{11!}{5! \cdot 6!} = 462$ You can take 462 different routes.

14. e.g., Let's assign people to the 5-person car first. Let J represent the number of ways to assign the people to this car:

$J = {}_{16}C_5$

$J = \frac{16!}{5!11!}$

$J = 4368$

Now there are 16 – 5 or 11 people left to assign to the remaining two vehicles. Let's assign people to the 4-person car next. Let K represent the number of ways to assign the people to this car:

$K = {}_{11}C_4$

$K = \frac{11!}{4!7!}$

$K = 330$

Now there are 11 – 4 or 7 people left to assign to the remaining vehicle. There is only 1 way to assign these people to the 7-passenger van because all of them are going to be assigned to it. Now let T represent the total number of assignments:

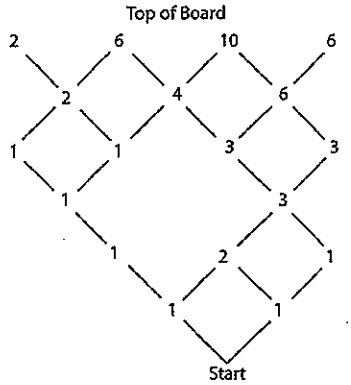
$T = J \cdot K \cdot 1$

$T = 4368 \cdot 330$

$T = 1\,441\,440$

There are 1 441 440 ways the 16 people can be assigned to the 3 vehicles.

15.



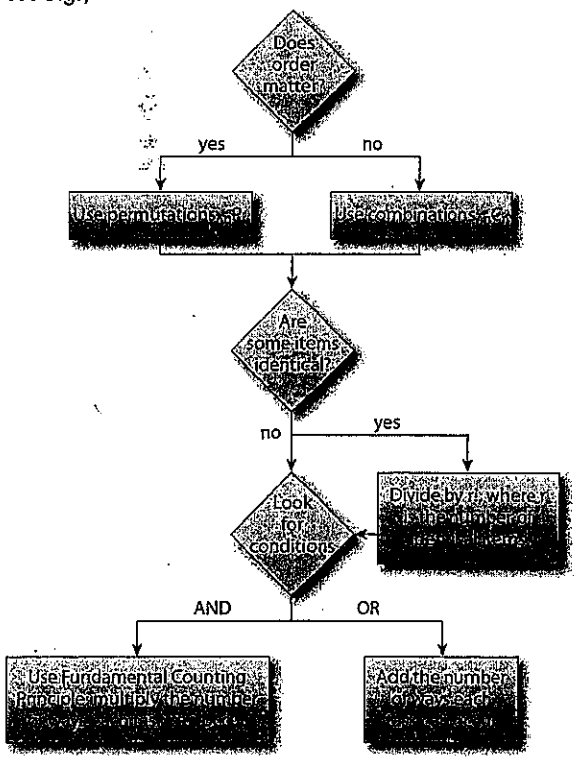
Number of Paths = 2 + 6 + 10 + 6
Number of Paths = 24
There are 24 paths that the red checker can follow.

16. Case 1: 0 hearts and 5 non-hearts: ${}_{13}C_0 \cdot {}_{39}C_5$
Case 2: 1 heart and 4 non-hearts: ${}_{13}C_1 \cdot {}_{39}C_4$
Case 3: 2 hearts and 3 non-hearts: ${}_{13}C_2 \cdot {}_{39}C_3$
Case 4: 3 hearts and 2 non-hearts: ${}_{13}C_3 \cdot {}_{39}C_2$
Let H represent the number of hands with at most 3 hearts:

$H = {}_{13}C_0 \cdot {}_{39}C_5 + {}_{13}C_1 \cdot {}_{39}C_4 + {}_{13}C_2 \cdot {}_{39}C_3 + {}_{13}C_3 \cdot {}_{39}C_2$
$H = 1 \cdot 575\,757 + 13 \cdot 82\,251 + 78 \cdot 9139 + 286 \cdot 741$
$H = 2\,569\,788$

There are 2 569 788 different five-card hands that contain at most three hearts that can be dealt.

17. e.g.,



18. Number of Total Outcomes:

$${}_{13}C_6 = \frac{13!}{6! \cdot 7!}$$

$${}_{13}C_6 = 1716$$

Number of Outcomes Where 3 Boys and 3 Girls Can Go:

$${}_6C_3 \cdot {}_7C_3 = \frac{6!}{3! \cdot 3!} \cdot \frac{7!}{3! \cdot 4!}$$

$${}_6C_3 \cdot {}_7C_3 = 700$$

Probability (P):

$$P = \frac{700}{1716} \times 100\%$$

$$P = 40.792...\%$$

There is about a 40.8% chance that there will be three boys and three girls on the trip.

19. e.g., If I have an A as the first letter, there are 4 possibilities for the second letter: A, L, S, or K.
If A is the second letter:
4 possibilities for the third letter: A, L, S, or K
Each one of these has 3 possibilities for the fourth letter. 4(3) = 12
If the second letter is L, S, or K:
3 possibilities for the third letter: A and 2 of L, S, and K (depending on which letter is second)
The A's have 3 possibilities for the fourth letter, and the other two letters have 2 possibilities for the fourth letter. 3 + 2(2) = 7
Total for all three second letters that are L, S, or K:
7(3) = 21
Total if A is the first letter:
21 + 12 = 33
Therefore, if the first letter is A, there are 33 possible arrangements.
If the first letter is L, S, or K, there are three possibilities for the second letter: A, and 2 of L, S, and K (the ones that are not the first letter).
If A is the second letter:
3 possibilities for the third letter: A, and 2 of L, S, and K
The A has 3 possibilities for the fourth letter and the other two letters have 2. 3 + 2(2) = 7
If A is not the second letter:
2 possibilities for the third letter
The A has 2 possibilities for the fourth letter and the other letter has 1. 2 + 1 = 3
Total for both second letters that are not A:
3(2) = 6
Total for one of three times where first letter is L, S, or K
7 + 6 = 13
Total when first letter is L, S, or K:
3(13) = 39
Total arrangements:
39 + 33 = 72
Therefore, 72 four-letter arrangements can be made using all of the letters in the word ALASKA.

20. If I have an O as the first letter, there are 4 possibilities for the second letter, each of which has 3 possibilities for the third letter. 4(3) = 12
Therefore, there are 12 possible arrangements when O is the first letter.
If the first letter is B, K, or S:
There are 3 possibilities for the second letter: O, and two of B, K, and S (the ones that are not the first letter). O has 3 possibilities for the third letter while the other 2 have 2. 3 + 2(2) = 7
Total if the first letter is B, K, or S:
3(7) = 21
Total arrangements:
21 + 12 = 33
Therefore, 33 three-letter arrangements can be made using all of the letters in the word BOOKS.

History Connection, page 128

A. Yes. Each number from 0 to 127 is assigned a different character or symbol on the keyboard. Since the numbers already have an established order, the characters and symbols assigned to these numbers do, as well.
B. Yes. Each number in ASCII (pronounced "askey") must be converted into a string of 0s and 1s to create the binary code, so order matters. Each 0 or 1 is associated with a position in the string. A different permutation of 0s and 1s represents a different number in the ASCII code system.
C. There are 128 numbers in ASCII that must be represented by a string of 0s and 1s. You need to determine the length of the string needed to create 128 different arrangements of 0s and 1s. You can begin by thinking about a string of length of 5.
A box diagram, ☐☐☐☐☐, can help you determine the number of ASCII numbers you can represent.
Within each box you can place a 0 or a 1. There are two choices for each box, since repetition of 0s and 1s is allowed. So for a string length of 5, there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$ or 32 ASCII numbers that can be represented. Obviously, the string must be longer for 128 numbers. If n represents the string length, and 128 numbers must be represented, then $2^n = 128$. By trial and error, $n = 7$.
A binary string of length 7 is needed to represent each ASCII code.

Chapter Self-Test, page 129

1. a) Let N represent the number of different serial numbers:
$N = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 3$
$N = 2\,028\,000$
There are 2 028 000 different serial numbers possible, if repetition of characters is allowed.

b) Let N represent the number of different serial numbers:
$N = 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 3$
$N = 1\,296\,000$
There are 1 296 000 different serial numbers possible, if no repetition is allowed.

2. Event A: Drawing a spade
Event B: Drawing a diamond
$n(A \cup B) = n(A) + n(B)$
$n(A \cup B) = 13 + 13$
$n(A \cup B) = 26$
Therefore, there are 26 ways to draw 1 card that is a spade or a diamond.

3. a) $n+9 \geq 0$
$n \geq -9$
$(n + 10)(n + 9)!$ is defined for $n \geq -9$, where $n \in I$.
$(n + 10)(n + 9)! = (n + 10)[(n + 9)(n + 8)...(3)(2)(1)]$
$(n + 10)(n + 9)! = (n + 10)(n + 9)(n + 8)...(3)(2)(1)$
$(n + 10)(n + 9)! = (n + 10)!$
b) $n-2 \geq 0$ AND $n \geq 0$
$n \geq 2$
$\frac{(n-2)!}{n!}$ is defined for $n \geq 2$, where $n \in I$.
$$\frac{(n-2)!}{n!} = \frac{(n-2)(n-3)...(3)(2)(1)}{n(n-1)(n-2)(n-3)...(3)(2)(1)}$$
$$\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$
$$\frac{(n-2)!}{n!} = \frac{1}{n^2-n}$$

4. a) $5! = 120$
Therefore, there are 120 different ways that the 5 cars can be parked side by side.
b) Let B represent the number of arrangements:
$B = {}_2P_2 \cdot {}_4P_4$
$B = 2! \cdot 4!$
$B = 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$B = 48$
Therefore, there are 48 different ways the cars can be parked so the 2 black cars are next to each other.

5. a) $${}_9C_4 = \frac{9!}{4!5!}$$
$${}_9C_4 = 126$$
There are 126 different four-book selections that can be made.

b) $_9P_4 = \frac{9!}{(9-4)!}$

$_9P_4 = \frac{9!}{5!}$

$_9P_4 = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!}$

$_9P_4 = 9\cdot 8\cdot 7\cdot 6$

$_9P_4 = 3024$

There are 3024 different four-book selections can be arranged in order of preference.

c) e.g., The order matters in part b). There are still 126 ways to choose the four books from the nine options, but there are also 4! = 24 ways to arrange the books.(126 · 24 = 3024)

6. $_nP_4 = 84(_nC_2)$

$\frac{n!}{(n-4)!} = 84\left[\frac{n!}{2!(n-2)!}\right]$

$\frac{n!}{(n-4)!} = \frac{42n!}{(n-2)!}$

$n(n-1)(n-2)(n-3) = 42n(n-1)$

$(n-2)(n-3) = 42$

$n^2 - 3n - 2n + 6 = 42$

$n^2 - 5n - 36 = 0$

$(n+4)(n-9) = 0$

$n+4=0$ or $n-9=0$

$n=-4$ $n=9$

Check $n = -4$

LS	RS
$_{-4}P_4$	$84(_{-4}C_2)$
$\frac{(-4)!}{(-4-4)!}$	$84\left[\frac{(-4)!}{2!(-4-2)!}\right]$
$\frac{(-4)!}{(-8)!}$ is undefined	

Check $n = 9$

LS	RS
$_9P_4$	$84(_9C_2)$
$\frac{9!}{(9-4)!}$	$84\left[\frac{9!}{2!(9-2)!}\right]$
$\frac{9!}{5!}$	$84\left[\frac{9!}{2!\cdot 7!}\right]$
$\frac{9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!}$	$84\left[\frac{9\cdot 8\cdot 7!}{2\cdot 1\cdot 7!}\right]$
$9\cdot 8\cdot 7\cdot 6$	$84\left[\frac{9\cdot 8}{2\cdot 1}\right]$
3024	$84[9\cdot 4]$
	3024

There is one solution, $n = 9$.

7. a) $_6C_2 \cdot {}_8C_3 = \frac{6!}{2!4!}\cdot\frac{8!}{3!5!}$

$_6C_2 \cdot {}_8C_3 = 840$

There are 840 different ways that a 5-person committee can be selected if there must be 2 boys and 3 girls.

b) Case 1: 2 boys and 3 girls:

$_6C_2 \cdot {}_8C_3 = \frac{6!}{2!4!}\cdot\frac{8!}{3!5!}$

$_6C_2 \cdot {}_8C_3 = 840$

Case 2: 3 boys and 2 girls:

$_6C_3 \cdot {}_8C_2 = \frac{6!}{3!3!}\cdot\frac{8!}{2!6!}$

$_6C_3 \cdot {}_8C_2 = 560$

Case 3: 4 boys and 1 girl:

$_6C_4 \cdot {}_8C_1 = \frac{6!}{4!2!}\cdot\frac{8!}{1!7!}$

$_6C_4 \cdot {}_8C_1 = 120$

Case 4: 5 boys and 0 girls:

$_6C_5 \cdot {}_8C_0 = \frac{6!}{5!1!}\cdot\frac{8!}{0!8!}$

$_6C_5 \cdot {}_8C_0 = 6$

Let C represent the number of 5-person committees with at least 2 boys:

$C = 840 + 560 + 120 + 6$

$C = 1526$

There are 1526 different ways that a 5-person committee can be selected if there must be at least 2 boys.

c) $_{12}C_3 = \frac{12!}{3!9!}$

$_{12}C_3 = 220$

There are 220 different ways that a 5-person committee can be selected if David and Susan must be on the committee.

d) Case 1: 2 boys and 3 girls

$${}_6C_2 \cdot {}_8C_3 = \frac{6!}{2!4!} \cdot \frac{8!}{3!5!}$$

$${}_6C_2 \cdot {}_8C_3 = 840$$

Case 2: 1 boy and 4 girls

$${}_6C_1 \cdot {}_8C_4 = \frac{6!}{1!5!} \cdot \frac{8!}{4!4!}$$

$${}_6C_1 \cdot {}_8C_4 = 420$$

Case 3: 0 boys and 5 girls

$${}_6C_0 \cdot {}_8C_5 = \frac{6!}{0!6!} \cdot \frac{8!}{5!3!}$$

$${}_6C_0 \cdot {}_8C_5 = 56$$

Let C represent the number of 5-person committees with more girls than boys:

$C = 840 + 420 + 56$

$C = 1316$

There are 1316 different ways that a 5-person committee can be selected if there must be more girls than boys.

8. $\frac{5!}{2!2!} = 30$

There are 30 different arrangements of the letters in the word TEETH.

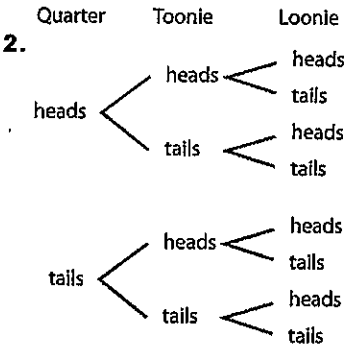
9. $5! \cdot 4! = 2880$

There are 2880 different arrangements possible.

Chapter Review, page 131

1. e.g., The Fundamental Counting Principle is used when a counting problem has different tasks related by the word AND. For example, you can use it to figure out how many ways you can roll a 3 with a die and draw a red card from a deck of cards.

2.



The tree diagram shows there are 8 possible ways that the three coins can land.

3. Let A represent the number of sets of answers:

$A = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

$A = 4^{10}$

$A = 1\,048\,576$

The student can give 1 048 576 different sets of answers.

4. a) $n+2 \geq 0$ AND $n \geq 0$

$n \geq -2$

$\frac{(n+2)!}{n!} = 20$ is defined for $n \geq 0$, where $n \in I$.

$$\frac{(n+2)!}{n!} = 20$$

$$\frac{(n+2)(n+1)(n)(n-1)...(3)(2)(1)}{(n)(n-1)...(3)(2)(1)} = 20$$

$$(n+2)(n+1) = 20$$

$$n^2 + n + 2n + 2 - 20 = 0$$

$$n^2 + 3n - 18 = 0$$

$$(n+6)(n-3) = 0$$

$n+6=0$ or $n-3=0$

$n=-6$ $\quad$ $n=3$

The root $n = -6$ is outside the restrictions on the variable in the equation, so it cannot be a solution. There is one solution, $n = 3$.

b) The simplified version of the equation is $\frac{(n+1)!}{(n-1)!} = 132$.

$n+1 \geq 0$ AND $n-1 \geq 0$

$n \geq -1$ $\quad$ $n \geq 1$

$\frac{(n+1)!}{(n-1)!} = 132$ is defined for $n \geq 1$, where $n \in I$.

$$\frac{(n+1)!}{(n-1)!} = 132$$

$$\frac{(n+1)(n)(n-1)(n-2)...(3)(2)(1)}{(n-1)(n-2)...(3)(2)(1)} = 132$$

$$(n+1)(n) = 132$$

$$n^2 + n = 132$$

$$n^2 + n - 132 = 0$$

$$(n+12)(n-11) = 0$$

$n+12=0$ or $n-11=0$

$n=-12$ $\quad$ $n=11$

The root $n = -12$ is outside the restrictions on the variable in the equation, so it cannot be a solution. There is one solution, $n = 11$.

5. e.g., ${}_6P_6$ has a larger value. e.g., I know because $\frac{8!}{6!}$ is the factorial expression for the permutation expression ${}_8P_2$. Here, I have more objects than for ${}_6P_6$, but I am not using all of them. This leads to fewer possible arrangements, or in other words, a lower value for $\frac{8!}{6!}$.

6. Let O represent the number of orders:
$O = 12!$
$O = 479\,001\,600$
There are 479 001 600 different orders in which the singers could perform the 12 songs.

7. ${}_{25}P_3 = \frac{25!}{22!}$
${}_{25}P_3 = 13\,800$
There are 13 800 different ways a director of education, a superintendent of curriculum, and a superintendent of finance can be selected.

8. a) ${}_{25}P_{10} = \frac{25!}{15!}$
${}_{25}P_{10} = 11\,861\,676\,288\,000$
${}_{25}P_{10} = 1.186...\times 10^{13}$
There are 11 861 676 288 000 or about 1.2×10^{13} different ways the test can be created if there are no conditions.

b) ${}_{23}P_8 = \frac{23!}{15!}$
${}_{23}P_8 = 19\,769\,460\,480$
${}_{23}P_8 = 1.976...\times 10^{10}$
There are 19 769 460 480 or about 2.0×10^{10} different ways the test can be created if the easiest question of the 25 is always first and the most difficult question is always last.

9. ${}_{52}P_5 = \frac{52!}{47!}$
${}_{52}P_5 = 311\,875\,200$
There are 311 875 200 different five-card arrangements possible.

10. a) Let A represent the number of arrangements:
$\frac{11!}{2!2!2!} = \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot2\cdot1\cdot2\cdot1}$
$\frac{11!}{2!2!2!} = 11\cdot10\cdot9\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$
$\frac{11!}{2!2!2!} = 4\,989\,600$
There are 4 989 600 different arrangements that are possible if all the letters are used.

b) Let A represent the number of arrangements:
$\frac{10!}{2!2!2!} = \frac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot2\cdot1\cdot2\cdot1}$
$\frac{10!}{2!2!2!} = 10\cdot9\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$
$\frac{10!}{2!2!2!} = 453\,600$
There are 453 600 different arrangements that are possible if all the letters are used, but each arrangement must begin with the C.

11. a) $\frac{14!}{2!3!4!5!} = 2522520$
There are 2 522 520 different ways Tina can stack the blocks in a single tower if there are no conditions.

b) $\frac{12!}{3!4!5!} = 27720$
There are 27 720 different ways Tina can stack the blocks in a single tower if there must be a yellow block at the bottom of the tower and a yellow block at the top.

12. ${}_{10}C_5 = \frac{10!}{5!5!}$ ${}_{11}C_7 = \frac{11!}{7!4!}$ ${}_{15}C_2 = \frac{15!}{2!13!}$
${}_{10}C_5 = 252$ ${}_{11}C_7 = 330$ ${}_{15}C_2 = 105$
Therefore, ${}_{11}C_7$ results in the greatest value.

13. ${}_{20}C_4 = \frac{20!}{4!16!}$
${}_{20}C_4 = 4845$
There are 4845 different selections of 4 books that Ruth can choose.

14. No. e.g., Each combination can be arranged in many different ways to make a permutation, so there are more permutations than combinations

15. a) ${}_{19}C_4 = \frac{19!}{4!15!}$
${}_{19}C_4 = 3876$
There are 3876 different ways that a committee of 4 people can be chosen if there are no conditions.

b) ${}_9C_2 \cdot {}_{10}C_2 = \frac{9!}{2!7!} \cdot \frac{10!}{2!8!}$
${}_9C_2 \cdot {}_{10}C_2 = 36 \cdot 45$
${}_9C_2 \cdot {}_{10}C_2 = 1620$
There are 1620 different ways that a committee of 4 people can be chosen if there must be an equal number of men and women on the committee.

c) ${}_{10}C_4 = \frac{10!}{4!6!}$
${}_{10}C_4 = 210$
There are 210 different ways that a committee of 4 people can be chosen if no men can be on the committee.

16. e.g., Let A represent the number of ways to assign teachers to the first group of 5:

$A = {}_{15}C_5$

$A = \frac{15!}{5!10!}$

$A = 3003$

Now there are 15 – 5 or 10 teachers left to assign. Let B represent the number of ways to assign the remaining teachers to the second group of 5:

$B = {}_{10}C_5$

$B = \frac{10!}{5!5!}$

$B = 252$

Now there are 10 – 5 or 5 teachers left to assign to the last group of 5. There is only 1 way that this can be done. Let T represent the total number of ways to assign the teachers:

$T = A \cdot B$

$T = 3003 \cdot 252$

$T = 756\,756$

There are 756 756 different ways 15 teachers can be divided into 3 groups of 5.

17. e.g., The first point can be joined with 11 more points to form straight lines. The second point can then be joined with 10 more points to form straight lines (since it was already joined with the first point). The third point can be joined with 9 more points to form straight lines (since it was already joined with the first two points). This pattern continues on until I get to the second-last point that can only be joined with the last point (since it was already joined with the other 10 points). The last point cannot be joined any further since it is already joined to every other point in the circle. Using this observed pattern, I can calculate the number of straight lines (L):

$L = 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$

$L = 66$

There are 66 different ways the points can be joined to form straight lines.

18. a) Since there is one more boy than there are girls, the line must always follow this pattern: BGBGBGBGBGBGB. Thus the boys are arranged in 7 positions, and the girls in 6 positions.

$7! \cdot 6! = 3\,628\,800$

There are 3 628 800 ways in which the children can be arranged in one row if the boys and girls must alternate positions.

b) Group the triplets as one. There are 3! ways in which the triplets can arrange themselves. Let B represent the number of different arrangements:

$B = 11! \cdot 3!$

$B = 39\,916\,800 \cdot 6$

$B = 239\,500\,800$

There are 239 500 800 ways in which the children can be arranged in one row if the triplets must stand next to each other.

19. Case 1: 2 face cards and 3 non-face cards: ${}_{12}C_2 \cdot {}_{40}C_3$
Case 2: 3 face cards and 2 non-face cards: ${}_{12}C_3 \cdot {}_{40}C_2$
Case 3: 4 face cards and 1 non-face card: ${}_{12}C_4 \cdot {}_{40}C_1$
Case 4: 5 face cards and 0 non-face cards: ${}_{12}C_5 \cdot {}_{40}C_0$
Let H represent the number of hands with at least 2 face cards:

$H = {}_{12}C_2 \cdot {}_{40}C_3 + {}_{12}C_3 \cdot {}_{40}C_2 + {}_{12}C_4 \cdot {}_{40}C_1 + {}_{12}C_5 \cdot {}_{40}C_0$

$H = 66 \cdot 9880 + 220 \cdot 780 + 495 \cdot 40 + 792 \cdot 1$

$H = 844\,272$

There are 844 272 different five-card hands with at least two face cards.

Chapter Task, page 133

A. Combinations. The order in which the dice are tossed does not matter (note that players toss all 8 dice simultaneously) nor does the way the dice are arranged when they land matter. What is important is the outcome of each toss—a combination of number of dice that land with the same side up AND number of dice that land with a different side up, for example, 7 dice land with the same face up AND 1 die with the opposite face up.

B. Each outcome can happen two ways. For example, 8 with the same side up could occur as 8 of the unmarked sides face up or 8 of the marked sides face up. That is why each calculation is the sum of two combination values:

8 dice land with the same side up

$= \binom{8}{8} + \binom{8}{8}$ or 1 + 1 or 2

7 dice land with the same side up

$= \binom{8}{7}\binom{1}{1} + \binom{8}{7}\binom{1}{1}$ or 8 + 8 or 16

6 dice land with the same side up

$= \binom{8}{6}\binom{2}{2} + \binom{8}{6}\binom{2}{2}$ or 28 + 28 or 56

5 dice land with the same side up

$= \binom{8}{5}\binom{3}{3} + \binom{8}{5}\binom{3}{3}$ or 56 + 56 or 112

4 dice land with the same side up

$= \binom{8}{4}\binom{4}{4} + \binom{8}{4}\binom{4}{4}$ or 70 + 70 or 140

3 dice land with the same side up

$= \binom{8}{3}\binom{5}{5} + \binom{8}{3}\binom{5}{5}$ or 56 + 56 or 112

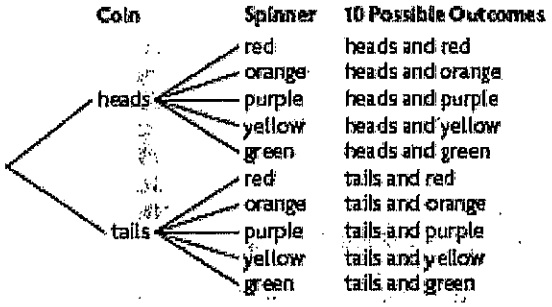
c. Yes, I think it is fair.

Outcome	Points	Number of Ways Outcome Can Occur
8 dice land same side up	10	2
7 dice land same side up	4	16
6 dice land same side up	2	56
5, 4, or 3 dice land same side up	0	364

The highest number of beans (10) is awarded for the outcome that can happen in the least number of ways, 8 dice landing same side up; 4 beans are awarded for the outcome that can happen in the second fewest number of ways, 7 dice landing same side up; 2 beans are awarded for the outcome that can happen in the third fewest number of ways, 6 dice landing same side up. The most likely outcomes of 5, 4, and 3 dice landing same side up all receive the lowest number of beans (0). So the point system rewards the least likely outcomes with the most beans and the most likely outcomes with the fewest beans.

Chapter 2 Diagnostic Test, TR page 121

1. a)



b) b) e.g., By looking at the tree diagram, there is one way he could flip a head and spin green, and ten possible outcomes.

$P(\text{heads and green}) = \frac{1}{10}$ or 10%.

2. a)

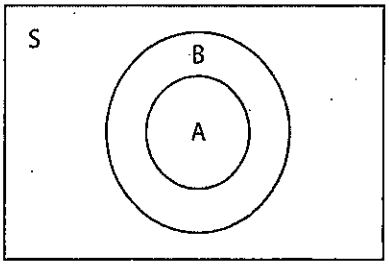
Child 1	Child 2	Child 3
B	B	B
B	G	B
B	B	G
G	B	B
G	G	B
G	B	G
B	G	G
G	G	G

b) There are two ways all three children will be the same gender, either all boys or all girls, and there are eight possible outcomes.

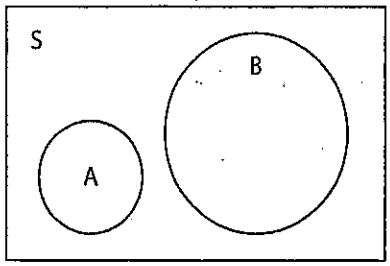
$P(\text{all boys or all girls}) = \frac{2}{8}$ or 25% chance, assuming that having a boy or girl is equally likely.

3. a) B' = {the set of elements not in B}
$B' = \{a, b, c, d, e, i, o, u\}$
b) $A \cup B$ = {the set of elements in A and B}
$A \cup B = \{a, b, c, d, e, i, o, u, x, y, z\}$
c) $A \cap B$ = {the set of elements in both A and B}
$A \cap B = \{y\}$

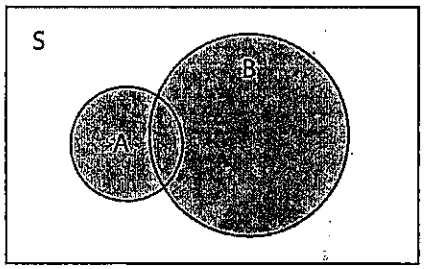
4. a)



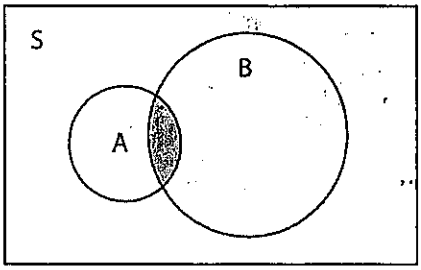
b)



c)



d)



5. e.g., Let x be the number of good singers without dancing skills. Let A be the set of singers and B the set of dancers.

$n(A) = x + 6$

$n(B) = 10 + 6$ or 16

$n(A \cap B) = 6$

$n(A \cup B) = 24$

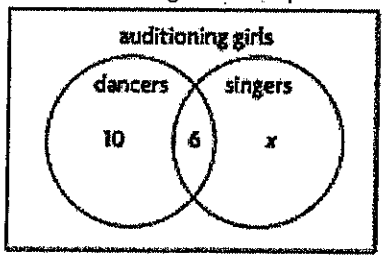
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$24 = x + 6 + 16 - 6$

$24 = x + 16$

$8 = x$

Use a Venn diagram to help solve the problem.



$10 + 6 + x = 24$

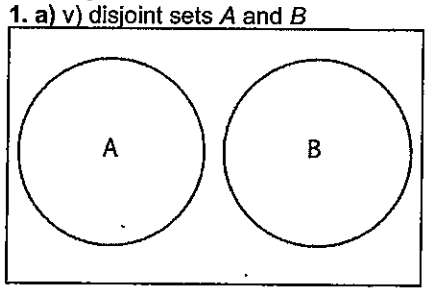
$16 + x = 24$

$x = 8$

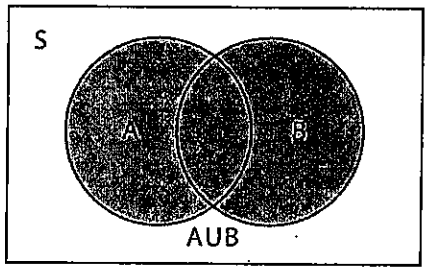
There were 8 girls who were good singers but not good dancers.

Review of Terms and Connections, TR page 124

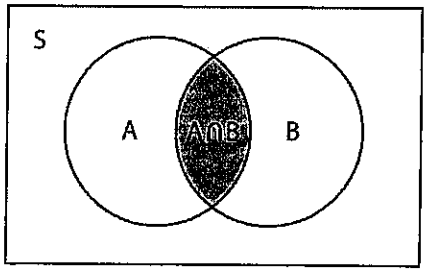
1. a) v) disjoint sets A and B



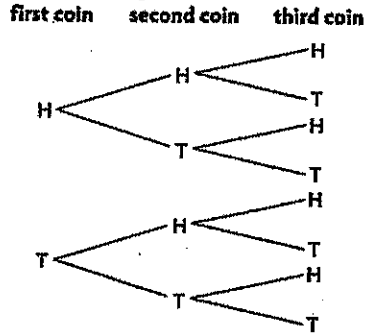
b) iii) the union of sets A and B



c) ii) the intersection of sets A and B



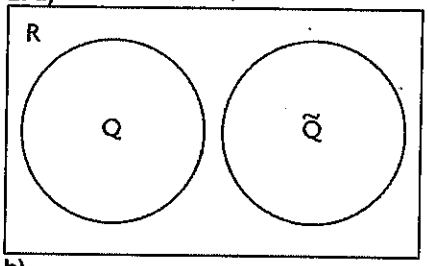
d) i) tree diagram



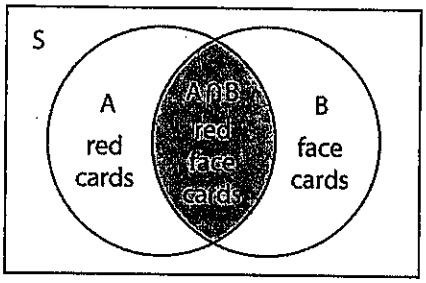
e) iv) outcome table

	Die 1					
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
Die 2 3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

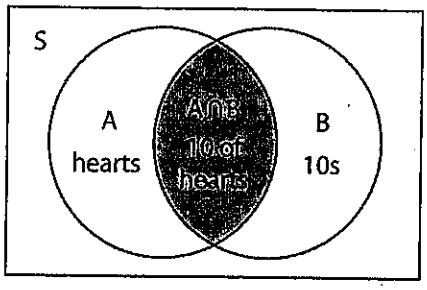
2. a)



b)



c)



3. a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A) = 12$, $n(B) = 9$, $n(A \cap B) = 5$

$n(A \cup B) = 12 + 9 - 5 = 16$

b) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A) = 23$, $n(B) = 16$, $n(A \cap B) = 1$

$n(A \cup B) = 23 + 16 - 1 = 38$

4. a) Let S represent the universal set of all students. Let A represent the students who attended the first school dance, and let B represent students who attended the second school dance. Then $n(A \cup B)$ is the number of students who went to one of the first two school dances.

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A) = 420$, $n(B) = 480$, $n(A \cap B) = 285$

$n(A \cup B) = 420 + 480 - 285 = 615$

615 students went to one of the first two school dances of the year.

b) $n(A \cup B)'$ is the number of students who did not attend either dance.

$n(A \cup B)' = S - n(A \cup B)$

$S = 1200$, $n(A \cup B) = 615$

$n(A \cup B)' = 1200 - 615 = 585$

585 students did not attend either dance.

5. a) B = {set of black face cards in a standard deck of playing cards} = {J♣, Q♣, K♣, J♠, Q♠, K♠}

b) D = {set of different three-digit numbers using the digits 1, 3, and 5} = {135, 153, 315, 351, 513, 531}

c) S = {set of all possible sums when a pair of dice is rolled} = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

d) T = {set of all the days of the week with names that begin with T} = {Tuesday, Thursday}

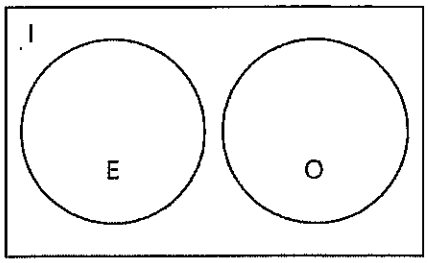
6. a) { }, {red}, {blue}, {red, blue}

b) { }, {2}, {4}, {6}, {8}, {2, 4}, {2, 6}, {2, 8}, {4, 6}, {4, 8}, {6, 8}, {2, 4, 6}, {2, 4, 8}, {4, 6, 8}, {2, 4, 6, 8}

c) { }, {April}, {May}, {June}, {April, May}, {April, June}, {May, June}, {April, May, June}

d) { }, {100}

7. e.g., I is the universal set of integers. E is the subset of even integers. O is the subset of odd integers.



8. S = {I, N, T, E, R, S, C, O}

a) A = {set of vowels in S}
A = {I, E, O}

b) B = {set of letters in SECTION}
B = {S, E, C, T, I, O, N}

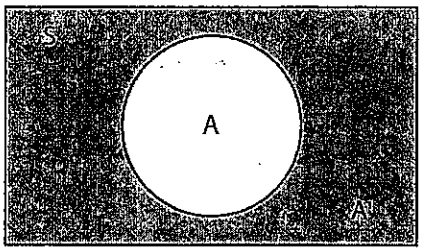
c) $A \cup B$
= {set of vowels in S and set of letters in SECTION}
= {S, E, C, T, I, O, N}

d) $A \cap B$ = {elements in both A and B}
$A \cap B$ = {I, E, O}

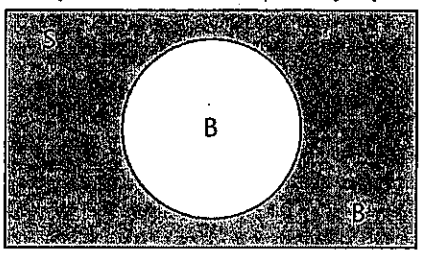
e) $A' \cup B'$ = {elements not in A and elements not in B}
$A' \cup B'$ = {N, T, R, S, C, E, I, O}

f) $A' \cap B'$ = {elements in neither A nor B} = {N, T, R}

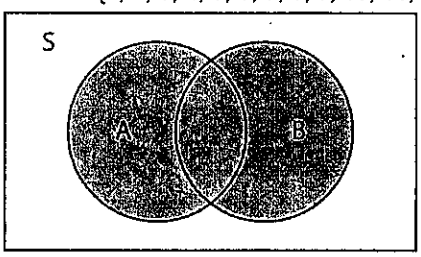
9. a) S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14}
A = {2, 4, 6, 8, 10}
B = {1, 3, 5, 7, 9, 11, 13}
A' = {the elements of S not in A}
A' = {1, 3, 5, 7, 9, 11, 12, 13, 14}



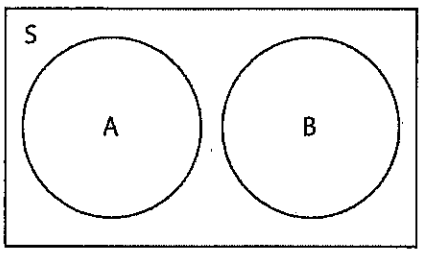
B' = {the elements of S not in B} = {2, 4, 6, 8, 10, 12, 14}



$A \cup B$ = {the elements of A and B}
= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13}



$A \cap B$ = {the elements in both A and B} = { }



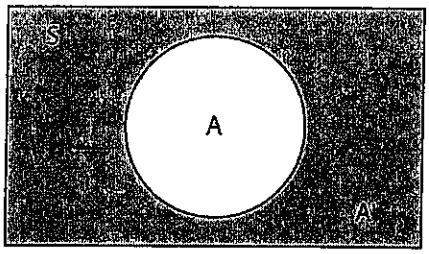
b) S = {A♠, 2♠, 3♠, 4♠, 5♠, 6♠, 7♠, 8♠, 9♠, 10♠, A♣, 2♣, 3♣, 4♣, 5♣, 6♣, 7♣, 8♣, 9♣, 10♣, A♥, 2♥, 3♥, 4♥, 5♥, 6♥, 7♥, 8♥, 9♥, 10♥, A♦, 2♦, 3♦, 4♦, 5♦, 6♦, 7♦, 8♦, 9♦, 10♦}

A = {3♠, 6♠, 9♠, 3♣, 6♣, 9♣, 3♥, 6♥, 9♥, 3♦, 6♦, 9♦}

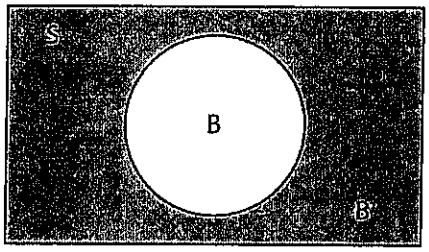
B = {2♠, 4♠, 6♠, 8♠, 10♠, 2♣, 4♣, 6♣, 8♣, 10♣, 2♥, 4♥, 6♥, 8♥, 10♥, 2♦, 4♦, 6♦, 8♦, 10♦}

A' = {the elements of S not in A}

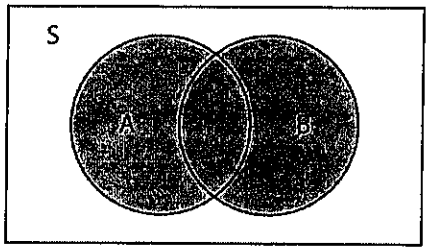
A' = {A♠, 2♠, 4♠, 5♠, 7♠, 8♠, 10♠, A♣, 2♣, 4♣, 5♣, 7♣, 8♣, 10♣, A♥, 2♥, 4♥, 5♥, 7♥, 8♥, 10♥, A♦, 2♦, 4♦, 5♦, 7♦, 8♦, 10♦}



B' = {the elements of S not in B} = {A♠, 3♠, 5♠, 7♠, 9♠, A♣, 3♣, 5♣, 7♣, 9♣, A♥, 3♥, 5♥, 7♥, 9♥, A♦, 3♦, 5♦, 7♦, 9♦}

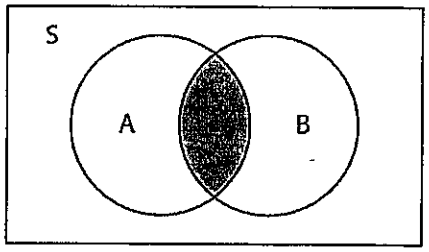


$A \cup B$ = {the elements of A and B} = {2♠, 3♠, 4♠, 6♠, 8♠, 9♠, 10♠, 2♣, 3♣, 4♣, 6♣, 8♣, 9♣, 10♣, 2♥, 3♥, 4♥, 6♥, 8♥, 9♥, 10♥, 2♦, 3♦, 4♦, 6♦, 8♦, 9♦, 10♦}



$A \cap B$ = {the elements in both A and B}

$A \cap B$ = {6♠, 6♣, 6♥, 6♦}



10. e.g.,
a) S is the universal set of all students in my math class.
A is the subset of students who are taller than 6 ft.
B is the subset of all students with black hair.
$A \cap B$ (shaded) is the subset of students who are taller than 6 ft and have black hair.
b) S is the universal set of all cards in a standard deck of playing cards.
A is the subset of all face cards.
B is the subset of all red cards.
$A \cup B$ (shaded) is the subset of all face cards and all red cards.

11. Let S represent the universal set of all first-year students.
Let C represent the students who take calculus.
Let A represent the students who take algebra. Then $n(C \cup A)'$ will be the number of first-year students who take neither calculus nor algebra.
$n(S) = 200$
$n(C) = 110$
$n(A) = 75$
$n(C \cap A) = 60$

$n(C \cup A) = n(C) + n(A) - n(C \cap A)$
$n(C \cup A) = 110 + 75 - 60$
$n(C \cup A) = 125$

$n(C \cup A)' = S - n(C \cup A)$
$n(C \cup A)' = 200 - 125$
$n(C \cup A)' = 75$
There are 75 first-year students who take neither calculus nor algebra.

Chapter 2 Test, TR page 134

1. e.g., The table shows the possible wins and losses of one of the teams. Shaded cells indicate games that would not actually be played, since one of the teams will have already won or lost two games. That means there are only six different outcomes.

Game 1	Game 2	Game 3	Outcomes
W	W	W	1 outcome
W	W	L	
W	L	W	1 outcome
W	L	L	1 outcome
L	W	W	1 outcome
L	W	L	1 outcome
L	L	W	1 outcome
L	L	L	

b) There are two different ways a team can lose exactly one game but still win the championship: WLW or LWW.

2. a) The sandwich has four elements: egg salad or chicken salad, lettuce or tomato, butter or mayonnaise, and whole wheat bun or whole grain bagel or sesame seed bun. Using the Fundamental Counting Principle:
$2 \cdot 2 \cdot 2 \cdot 3 = 24$ ways to make a sandwich. I assumed that the sandwich has exactly one item from each of set of choices.
b) For each of the four digits, there are six options and repetition is allowed. Using the Fundamental Counting Principle:
$6 \cdot 6 \cdot 6 \cdot 6 = 1296$ ways to make a password. I assumed starting the password with a 0 was allowed.
c) There are 13 hearts and 13 clubs in a standard deck of 52 cards, and the sets of hearts and clubs are disjoint. There are $13 + 13 = 26$ ways to draw a heart or club. I made no assumptions.

d) There are five letters in TEETH with both T and E repeating twice. There are $\frac{5!}{2!2!} = 30$ arrangements of the letters. I made no assumptions.
e) There are 25 different toppings, and Jim must choose 3. Order does not matter. Jim can order $\binom{25}{3} = 2300$ different pizzas. I assumed he would choose exactly three toppings and each would be different.

3. a) Since order does not matter:

$$\binom{10}{5} = \frac{10!}{5!(10-5)!}$$

$$\binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5!5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$\binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$\binom{10}{5} = \frac{30\ 240}{120}$$

$$\binom{10}{5} = 252$$

There are 252 selections that can be made.
b) Since order matters:

$${}_{10}P_5 = \frac{10!}{(10-5)!}$$

$${}_{10}P_5 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5!}$$

$${}_{10}P_5 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$$

$${}_{10}P_5 = 30\ 240$$

There are 30 240 selections if the selections are ordered by preference.
c) e.g., Order does not matter in part a) but it does matter in part b), so part a) involves combinations, whereas part b) involves permutations.
d) e.g., The answer to part a) is 5! or 120 times smaller than the answer to part b). This is because the 30 240 five-novel selections from part b) must be divided by 5! to eliminate combinations that are the same, because order does not matter.

4. To solve the problem, look at the walk in three sections. In the first section, Maria has to walk three blocks east and three blocks south for a total of six blocks; in the second section she has to walk one block east and one block south for a total of two blocks; in the third section she has to walk two blocks east and two blocks south for a total of four blocks.
First section:

$$\frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!3 \cdot 2 \cdot 1}$$

$$\frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$$

$$\frac{6!}{3!3!} = 5 \cdot 4$$

$$\frac{6!}{3!3!} = 20$$

Second section:

$$\frac{2!}{1!1!} = \frac{2 \cdot 1}{1 \cdot 1}$$

$$\frac{2!}{1!1!} = 2$$

Third section:

$$\frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2!}{2!2 \cdot 1}$$

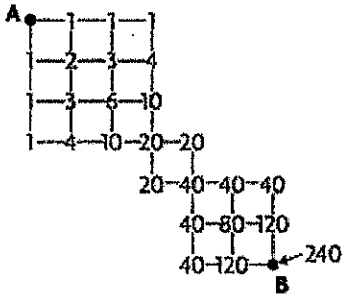
$$\frac{4!}{2!2!} = \frac{4 \cdot 3}{2}$$

$$\frac{4!}{2!2!} = 2 \cdot 3$$

$$\frac{4!}{2!2!} = 6$$

Using the Fundamental Counting Principle:
$20 \cdot 2 \cdot 6 = 240$
She can walk 240 different ways.

This problem can also be solved using a pathway diagram:



5. a) Order does not matter, so this is a combination problem. There are (5 2) ways to choose two boys from five, and (6 2) ways to choose two girls from six.

$$\binom{5}{2}\binom{6}{2}=\frac{5!}{(5-2)!2!}\cdot\frac{6!}{(6-2)!2!}$$

$$\binom{5}{2}\binom{6}{2}=\frac{5!}{3!2!}\cdot\frac{6!}{4!2!}$$

$$\binom{5}{2}\binom{6}{2}=\frac{5\cdot4\cdot3!}{3!2}\cdot\frac{6\cdot5\cdot4!}{4!2}$$

$$\binom{5}{2}\binom{6}{2}=\frac{5\cdot4}{2}\cdot\frac{6\cdot5}{2}$$

$$\binom{5}{2}\binom{6}{2}=10\cdot15$$

$$\binom{5}{2}\binom{6}{2}=150$$

There are 150 ways to choose a committee with two boys and two girls.

b) At least two girls means two, three, or four girls:

$$\binom{5}{2}\binom{6}{2}+\binom{5}{1}\binom{6}{3}+\binom{5}{0}\binom{6}{4}$$

$$=150+\left(\frac{5!}{(5-1)!1!}\cdot\frac{6!}{(6-3)!3!}\right)+\left(\frac{5!}{5!}\cdot\frac{6!}{(6-4)!4!}\right)$$

$$=150+\left(\frac{5!}{4!1!}\cdot\frac{6!}{3!3!}\right)+\left(1\cdot\frac{6!}{2!4!}\right)$$

$$=150+\left(5\cdot\frac{6\cdot5\cdot4}{3\cdot2}\right)+\left(1\cdot\frac{6\cdot5}{2}\right)$$

$$=150+(5\cdot20)+(1\cdot15)$$

$$=150+100+15$$

$$=265$$

c) If Jim and Nanci must be on the committee, there are nine people left for the remaining two positions:

$$\binom{1}{1}\binom{1}{1}\binom{9}{2}=1\cdot1\cdot\frac{9!}{(9-2)!2!}$$

$$\binom{1}{1}\binom{1}{1}\binom{9}{2}=1\cdot1\cdot\frac{9!}{7!2!}$$

$$\binom{1}{1}\binom{1}{1}\binom{9}{2}=1\cdot1\cdot\frac{9\cdot8\cdot7!}{7!2!}$$

$$\binom{1}{1}\binom{1}{1}\binom{9}{2}=1\cdot1\cdot\frac{9\cdot8}{2}$$

$$\binom{1}{1}\binom{1}{1}\binom{9}{2}=36$$

There are 36 committees that can be made if Jim and Nanci must be on the committee.

d) More boys than girls means three or four boys:

$$\binom{5}{3}\binom{6}{1}+\binom{5}{4}\binom{6}{0}=\frac{5!}{(5-3)!3!}\cdot\frac{6!}{(6-1)!1!}+\frac{5!}{(5-4)!4!}\cdot1$$

$$\binom{5}{3}\binom{6}{1}+\binom{5}{4}\binom{6}{0}=\frac{5!}{2!3!}\cdot\frac{6!}{5!1!}+\frac{5!}{1!4!}\cdot1$$

$$\binom{5}{3}\binom{6}{1}+\binom{5}{4}\binom{6}{0}=\frac{5\cdot4\cdot3!}{2!3!}\cdot\frac{6\cdot5!}{5!1!}+\frac{5\cdot4!}{1!4!}\cdot1$$

$$\binom{5}{3}\binom{6}{1}+\binom{5}{4}\binom{6}{0}=\frac{5\cdot4}{2}\cdot\frac{6}{1}+\frac{5}{1}\cdot1$$

$$\binom{5}{3}\binom{6}{1}+\binom{5}{4}\binom{6}{0}=10\cdot6+5\cdot1$$

$$\binom{5}{3}\binom{6}{1}+\binom{5}{4}\binom{6}{0}=60+5$$

$$\binom{5}{3}\binom{6}{1}+\binom{5}{4}\binom{6}{0}=65$$

There are 65 committees that can be made with more boys than girls.

6. a)

$${}_nP_4=60({}_nC_2)$$

$$\frac{n!}{(n-4)!}=60\left(\frac{n!}{2!(n-2)!}\right)$$

$$n(n-1)(n-2)(n-3)=60\left(\frac{n(n-1)}{2}\right)$$

$$(n-2)(n-3)=30$$

$$n^2-5n-24=0$$

$$(n-8)(n+3)=0$$

$$n=8 \text{ or } n=-3$$

$n=-3$ is extraneous

$n=8$

b) $n=8$, $n\geq4$

${}_nP_4=60({}_nC_2)$, so $n\geq4$ and $n\geq2$, so $n\geq4$

7. If you place the T and K as required, this can happen only one way for each position. The remaining seven letters can then be arranged in between, keeping in mind that there are repeated letters: two A's, two S's, and two O's:

$$1\cdot1\cdot\left(\frac{7!}{2!2!2!}\right)=1\cdot1\cdot\left(\frac{7\cdot6\cdot5\cdot4\cdot3\cdot2!}{2!2\cdot2}\right)$$

$$1\cdot1\cdot\left(\frac{7!}{2!2!2!}\right)=1\cdot1\cdot\left(\frac{7\cdot6\cdot5\cdot4\cdot3}{2\cdot2}\right)$$

$$1\cdot1\cdot\left(\frac{7!}{2!2!2!}\right)=1\cdot1\cdot\left(\frac{7\cdot6\cdot5\cdot4\cdot3}{4}\right)$$

$$1\cdot1\cdot\left(\frac{7!}{2!2!2!}\right)=1\cdot1\cdot(7\cdot6\cdot5\cdot4\cdot3)$$

$$1\cdot1\cdot\left(\frac{7!}{2!2!2!}\right)=630$$

There are 630 different ways to arrange the letters with the given conditions.